Connections between centrality and local monotonicity of certain functions on C^*-algebras

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Motivation

- Ogasawara[1] (1955): a C^*-algebra $\mathcal{A}$ is commutative if and only if the map $x \mapsto x^2$ is monotonic increasing on the set of the positive elements of $\mathcal{A}$
- Pedersen[2] (1979): for any $p \in (1, \infty)$, the map $x \mapsto x^2$ may be replaced by $x \mapsto x^p$ in the above theorem
- Wu[3] (2001): Ogasawara's result remains true when $x \mapsto x^2$ is replaced by $x \mapsto e^x$

[1] T. Ogasawara, *A theorem on operator algebras,* J. Sci. Hiroshima Univ. Ser. A. **18** (1955), 307-309.

[2] G.K. Pedersen, *C^*-Algebras and Their Automorphism Groups,* London Mathematical Society Monographs, 14, Academic Press, Inc., London-New York, 1979.

[3] W. Wu, *An order characterization of commutativity for C^*-algebras,* Proc. Amer. Math. Soc. **129** (2001), 983–987.

Motivation

- Ji and Tomiyama[4] (2003): let f be a continuous function on the positive axis which is monotonic increasing but is not matrix monotone of order 2. Then $\mathcal{A}$ is commutative if and only if f is monotone increasing on the positive cone of $\mathcal{A}$.
- Molnár[5] (2016): a positive element $x \in \mathcal{A}$ is central if and only if $x \leq y$ implies $e^x \leq e^y$
- Observe that Wu's theorem is an immediate consequence of Molnár's result
- Our goal is to provide "local" versions of the theorems of Ogasawara, Pedersen, Ji and Tomiyama

[4] G. Ji and J. Tomiyama, *On characterizations of commutativity of C^*-algebras,* Proc. Amer. Math. Soc. **131** (2003), 3845-3849.

[5] L. Molnár, *A characterization of central elements in C^*-algebras,* Bull. Austral. Math. Soc. **95** (2017), 138-143.

Basic notions

- the symbol $\mathcal{A}$ stands for a unital C^*-algebra and I denotes its unit
- the spectrum of an element $A \in \mathcal{A}$ is denoted by $\sigma(A)$ and it is defined by

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is not invertible}\}$$

- $\mathcal{A}_s$ stands for the set of the self-adjoint elements of $\mathcal{A}$, and we say that $A \in \mathcal{A}_s$ is positive, if $\sigma(A) \subset [0, \infty)$
- the partial order $\leq$ on $\mathcal{A}_s$ is defined as follows: for any self-adjoint elements A and B, we have $A \leq B$ if and only if $B - A$ is positive
- denote by $\mathcal{A}_+$ (resp. $\mathcal{A}_+^{-1}$) the set of all positive (resp. positive invertible) elements of $\mathcal{A}$

The main theorem

Theorem (V.)

Let $I = (\gamma, \infty)$ *for some* $\gamma \in \mathbb{R} \cup \{-\infty\}$ *and let* $f \in C^1(I)$ *such that*

(i) $f'(x) > 0 \qquad x \in I,$

(ii) $x < y \Rightarrow f'(x) < f'(y) \qquad x, y, \in I,$

(iii) $\log\left(f'\left(tx + (1-t)y\right)\right) \geq$
$t \log f'(x) + (1-t) \log f'(y) \qquad x, y, \in I, \ t \in [0,1].$

Let $\mathcal{A}$ *be a unital* C^**-algebra and let* $a \in \mathcal{A}$ *be a self-adjoint element with* $\sigma(a) \subset I$*. The followings are equivalent.*

(1) a *is central, that is,* $ab = ba \qquad b \in \mathcal{A},$

(2) f *is locally monotone at the point* a*, that is,*
$a \leq b \Rightarrow f(a) \leq f(b) \quad b \in \mathcal{A}_s.$

Examples

Example

Some intervals and functions satisfying the conditions given in the Theorem.

- $I = (0, \infty)$, $f(x) = x^p \qquad p > 1$,
- $I = (-\infty, \infty)$, $f(x) = e^x$.

Notation. If φ and ψ are elements of some Hilbert space $\mathcal{H}$, then the symbol $\varphi \otimes \psi$ denotes the linear map $\mathcal{H} \ni \xi \mapsto \langle \xi, \psi \rangle \varphi \in \mathcal{H}$. The inner product is linear in its first variable.

The following Lemma is a key step of the proof of the Theorem.

Outline of the proof

Lemma

Suppose that $I = (\gamma, \infty)$ *for some* $\gamma \in \mathbb{R} \cup \{-\infty\}$ *and* $f \in C^1(I)$ *satisfies the conditions* (i), (ii) *and* (iii) *given in the Theorem. Let* $\mathcal{K}$ *be a two-dimensional Hilbert space, let* $\{u, v\} \subset \mathcal{K}$ *be an orthonormal basis. Let* $x, y \in I$ *and set* $A := xu \otimes u + yv \otimes v$. *The followings are equivalent.*

(I) $x \neq y$,

(II) *there exist* $\lambda, \mu \in \mathbb{C}$ *(with* $|\lambda|^2 + |\mu|^2 = 1$*) and* $t_0 > 0$ *such that using the notation* $B = (u + v) \otimes (u + v)$ *and* $w = \lambda u + \mu v$ *we have*

$$\langle f(A)w,\, w\rangle - \langle f\left(A + t_0 B\right) w,\, w\rangle > 0.$$

Outline of the proof

Notation. For any fixed interval $I = (\gamma, \infty)$ and function $f \in C^1(I)$ with the properties (i), (ii) and (iii), and different numbers $x, y \in I$, the above Lemma provides a positive number $\langle f(A)w,\, w\rangle - \langle f\left(A + t_0 B\right) w,\, w\rangle$. Let us introduce

$$\delta_{I,f,x,y} := \langle f(A)w,\, w\rangle - \langle f\left(A + t_0 B\right) w,\, w\rangle .$$

Proof of the Lemma.

- the direction (II) $\Rightarrow$ (I) is easy to see (by contraposition)
- to verify the direction (I) $\Rightarrow$ (II) we recall the following useful formula for the derivative of a matrix function[6]

[6]F. Hiai and D. Petz, *Introduction to Matrix Analysis and Applications*, Hindustan Book Agency and Springer Verlag (2014)

Outline of the proof

- if $A = xu \otimes u + yv \otimes v$, then for any self-adjoint $C \in \mathcal{B}(\mathcal{K})$ we have

$$\lim_{t \to 0} \frac{1}{t} \left(f\left(A + tC\right) - f(A) \right)$$

$$= f'(x) \left\langle Cu, u \right\rangle u \otimes u + \frac{f(x) - f(y)}{x - y} \left\langle Cv, u \right\rangle u \otimes v$$

$$+ \frac{f(y) - f(x)}{y - x} \left\langle Cu, v \right\rangle v \otimes u + f'(y) \left\langle Cv, v \right\rangle v \otimes v.$$

- matrix formalism: $[A] = \mathrm{diag}(x, y)$ and

$$\left[\lim_{t \to 0} \frac{1}{t} \left(f\left(A + tC\right) - f(A) \right) \right] = \begin{bmatrix} f'(x) & \frac{f(x)-f(y)}{x-y} \\ \frac{f(y)-f(x)}{y-x} & f'(y) \end{bmatrix} \circ [C]$$

Outline of the proof

- in particular, for $B = (u+v) \otimes (u+v)$ we have

$$L := \lim_{t \to 0} \frac{1}{t} \left(f(A+tB) - f(A) \right) = f'(x) u \otimes u + \frac{f(x)-f(y)}{x-y} u \otimes v + \\ + \frac{f(y)-f(x)}{y-x} v \otimes u + f'(y) v \otimes v. \qquad (1)$$

- the determinant of the matrix

$$[L] = \begin{bmatrix} f'(x) & \frac{f(x)-f(y)}{x-y} \\ \frac{f(y)-f(x)}{y-x} & f'(y) \end{bmatrix}$$

is negative as $\mathrm{Det}[L] < 0 \Leftrightarrow f'(x) f'(y) < \left(\frac{f(x)-f(y)}{x-y} \right)^2 \Leftrightarrow$

$$\Leftrightarrow \log f'(x) + \log f'(y) < 2 \log \left(\int_{t=0}^{1} f'(tx + (1-t)y) \, \mathrm{d}t \right)$$

Outline of the proof

- this latter inequality is true as

$$\log f'(x) + \log f'(y) = 2 \cdot \int_{t=0}^{1} t \log f'(x) + (1-t) \log f'(y) \mathrm{d}t$$

$$\leq 2 \int_{t=0}^{1} \log \left(f' \left(tx + (1-t)y \right) \right) \mathrm{d}t$$

$$< 2 \log \left(\int_{t=0}^{1} f' \left(tx + (1-t)y \right) \mathrm{d}t \right)$$

- in the above computation, the first inequality holds because of the log-concavity of f' and the second (strict) inequality holds because the logarithm function is strictly concave and f' is strictly monotone increasing

Outline of the proof

- so, the operator L (defined in eq. (1)) has a negative eigenvalue, that is, there exist $\lambda, \mu \in \mathbb{C}$ (with $|\lambda|^2 + |\mu|^2 = 1$) such that with $w = \lambda u + \mu v$ we have

$$\langle Lw, w \rangle = \left\langle \lim_{t \to 0} \frac{1}{t} \left(f\left(A + tB\right) - f(A) \right) w, w \right\rangle < 0$$

- therefore,

$$\lim_{t \to 0} \frac{1}{t} \left(\langle f\left(A + tB\right) w, w \rangle - \langle f(A) w, w \rangle \right) < 0,$$

and so there exists some $t_0 > 0$ such that
$0 < \langle f(A)w, w \rangle - \langle f\left(A + t_0 B\right) w, w \rangle$

Outline of the proof

The proof of the Theorem.

- the direction (1) $\Rightarrow$ (2) is easy to verify
- to see the contrary, assume that $a \in \mathcal{A}_s$, $\sigma(a) \subset I$ and $aa' - a'a \neq 0$ for some $a' \in \mathcal{A}$
- then there exists an irreducible representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\pi\left(aa' - a'a\right) \neq 0$, that is, $\pi(a)\pi\left(a'\right) \neq \pi\left(a'\right)\pi(a)$
- let us fix this irreducible representation π once and for all
- so, $\pi(a)$ is a non-central self-adjoint (and hence normal) element of $\mathcal{B}(\mathcal{H})$ with $\sigma\left(\pi(a)\right) \subset I$ (as a representation do not increase the spectrum)
- by the non-centrality, $\sigma\left(\pi(a)\right)$ has at least two elements, and by the normality, every element of $\sigma\left(\pi(a)\right)$ is an approximate eigenvalue

Outline of the proof

- let x and y be two different elements of $\sigma(\pi(a))$, and let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ and $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ satisfy

$$\lim_{n\to\infty} \pi(A)u_n - xu_n = 0, \ \lim_{n\to\infty} \pi(A)v_n - yv_n = 0,$$

  and $\langle u_m, v_n\rangle = 0 \quad m, n \in \mathbb{N}$ (as $x \neq y$, the approximate eigenvetors can be chosen to be orthogonal)
- set $\mathcal{K}_n := \mathrm{span}\{u_n, v_n\}$ and let E_n be the orthoprojection onto the closed subspace $\mathcal{K}_n^{\perp} \subset \mathcal{H}$
- let

$$\psi_n(a) := xu_n \otimes u_n + yv_n \otimes v_n + E_n\pi(a)E_n$$

- a direct computation shows that

$$\lim_{n\to\infty} \psi_n(a) = \pi(a)$$

  in the operator norm topology

Outline of the proof

- we have fixed I, f, x and y
- by the Lemma, we have $\lambda, \mu \in \mathbb{C}$ (with $|\lambda|^2 + |\mu|^2 = 1$) and $t_0 > 0$ such that using the notation $B_n := (u_n + v_n) \otimes (u_n + v_n)$ and $w_n := \lambda u_n + \mu v_n$, we have

$$\langle f\left(\psi_n(a)\right) w_n,\ w_n\rangle - \langle f\left(\psi_n(a) + t_0 B_n\right) w_n,\ w_n\rangle = \delta_{I,f,x,y} > 0 \tag{2}$$

  for any $n \in \mathbb{N}$
- that is, the left hand side of (2) is independent of n

- for any n, the operator B_n is a self-adjoint element of $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}_n$ is a finite dimensional subspace of $\mathcal{H}$, hence by Kadison's transitivity theorem, there exists a self-adjoint $b_n \in \mathcal{A}$ such that
$$\pi\left(b_n\right)_{|\mathcal{K}_n} = B_{n|\mathcal{K}_n}$$
- observe that $B_n\mathcal{K}_n \subseteq \mathcal{K}_n$ and so $\pi\left(b_n\right)\mathcal{K}_n \subseteq \mathcal{K}_n$
- on the other hand, $\pi\left(b_n\right)$ is self-adjoint as b_n is self-adjoint, hence it follows that $\pi\left(b_n\right)\mathcal{K}_n^{\perp} \subseteq \mathcal{K}_n^{\perp}$
- therefore, the fact $B_n = \frac{1}{2}B_n^2$ implies that

$$\pi\left(\frac{1}{2}b_n^2\right)_{|\mathcal{K}_n} = \left(\frac{1}{2}\pi\left(b_n\right)^2\right)_{|\mathcal{K}_n}$$

$$= \frac{1}{2}\left(\pi\left(b_n\right)_{|\mathcal{K}_n}\right)^2 = \frac{1}{2}B^2_{n|\mathcal{K}_n} = B_{n|\mathcal{K}_n}$$

- so we can rewrite (2) as

$$\langle f\left(\psi_n(a)\right) w_n,\ w_n\rangle - \left\langle f\left(\psi_n(a) + t_0\pi\left(\frac{1}{2}b_n^2\right)\right) w_n,\ w_n\right\rangle = \delta_{I,f,x,y} \tag{3}$$

- a standard continuity argument — which is based on the fact that $\psi_n(a)$ tends to $\pi(a)$ in the operator norm topology,— shows that

$$\lim_{n\to\infty} \|f\left(\psi_n(a)\right) - f\left(\pi(a)\right)\| = 0$$

and

$$\lim_{n\to\infty} \left\| f\left(\psi_n(a) + t_0\pi\left(\frac{1}{2}b_n^2\right)\right) - f\left(\pi(a) + t_0\pi\left(\frac{1}{2}b_n^2\right)\right)\right\| = 0.$$

Outline of the proof

- so for n large enough we have

$$\|f(\psi_n(a)) - f(\pi(a))\| < \frac{1}{4}\delta_{I,f,x,y}$$

and

$$\left\| f\left(\psi_n(a) + t_0\pi\left(\frac{1}{2}b_n^2\right)\right) - f\left(\pi\left(a + \frac{t_0}{2}b_n^2\right)\right)\right\| < \frac{1}{4}\delta_{I,f,x,y}.$$

- therefore, by (3), for n large enough the inequality

$$\langle f(\pi(a))\, w_n,\, w_n\rangle - \left\langle f\left(\pi\left(a + \frac{t_0}{2}b_n^2\right)\right) w_n,\, w_n\right\rangle > \frac{1}{2}\delta_{I,f,x,y} > 0 \tag{4}$$

holds

Outline of the proof

- in other words,

$$f\left(\pi(a)\right) \not\leq f\left(\pi\left(a+\frac{t_0}{2}b_n^2\right)\right)$$

or equivalently,

$$\pi\left(f(a)\right) \not\leq \pi\left(f\left(a+\frac{t_0}{2}b_n^2\right)\right)$$

- any representation of a C^*-algebra preserves the semidefinite order, hence this means that

$$f(a) \not\leq f\left(a+\frac{t_0}{2}b_n^2\right),$$

despite the fact that $a \leq a+\frac{t_0}{2}b_n^2$ □

A final remark

Remark

Note that our theorem generalizes Molnár's result, and — as every "local" theorem easily implies its "global" counterpart — we recover the theorems of Ogasawara, Pedersen, and Wu, as well.

Thank you for your attention!