On lower bounds for C_0-semigroups

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Trivial bounds

For $f \in L^1(\mathbb{R})$ define its Fourier transform by

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it\xi} f(t)\, dt,$$

and for $f \in L^1(0, 2\pi)$ define its Fourier coefficients (transform) by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(t)\, dt, \qquad n \in \mathbb{Z}.$$

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By the Riemann-Lebesgue Lemma:

$$\hat{f} \in C_0(\mathbb{R}), \qquad (\hat{f}(n))_{n\in\mathbb{Z}} \in c_0(\mathbb{Z}).$$

From Plancherel's (Parseval) theorem:

$\hat{f} \in L^2(\mathbb{R})$ jeśli $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $(\hat{f}(n))_{n\in\mathbb{Z}} \in l_2$ dla $f \in L^2(0, 2\pi))$.

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OUR AIM: Get the 'answer' in the framework of weak orbits of C_0-semigroups.

Fourier transforms of integrable functions

Theorem (Kolmogorov-Titchmarsh, 1920s)

1. Given $c = (c(n))_{n\in\mathbb{Z}} \in c_0(\mathbb{Z})$ there exists $f \in L^1(0, 2\pi)$ such that

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2. Given $c \in C_0(\mathbb{R})$ *there exists* $f \in L^1(\mathbb{R})$ *such that*

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Generalization[Curtis, Figa-Talamanca, 1966]: The same result is true in the context of locally compact abelian groups.

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1. Given $\{c_n\}_{n\in\mathbb{Z}} \in l_2(\mathbb{Z})$ there exists a 2π-periodic function $f \in C([0, 2\pi])$ such that

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2. Given $c \in L^2(\mathbb{R})$ there exists a function $f \in L^2(\mathbb{R}) \cap C_0(\mathbb{R})$ such that

$$|\hat{f}(\xi)| \geq |c(\xi)| \qquad \text{for almost every } \xi.$$

Many other settings !!!

Abstract setting

Theorem (K. Ball, Inventiones M. 1991, BLMS 1994)

1. If $\{x_n^* : n \in \mathbb{Z}\}$ *is a sequence of bounded linear functionals of norm* 1 *on a Banach space* X *and* $a = (a_n)_{n\in\mathbb{Z}} \in l_1(\mathbb{Z}), \|a\|_{l_1} < 1$, *then there exists* $x \in X$, $\|x\| \le 1$, *such that*

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2. If $\{x_n : n \in \mathbb{Z}\}$ is a sequence of elements of norm 1 in a Hilbert space H and $(a_n)_{n\in\mathbb{Z}} \in l_2(\mathbb{Z}), \|a\|_{l_2} < 1$, then there exists $x \in H$, $\|x\| \le 1$, such that

$$|(x_n, x)| \ge |a_n|, \qquad n \in \mathbb{Z}.$$

Implication for the Fourier transform:

Define the bounded linear functionals $x_n^*, n \in \mathbb{Z}$, on $L^1(\mathbb{R})$ by

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Then $\|x_n^*\| = 1, n \in \mathbb{Z}$, and for every $a = (a_n)_{n\in\mathbb{Z}} \in l_1(\mathbb{Z}), \|a\|_{l_1} < 1$, there exists $x \in L^1(\mathbb{R})$, $\|x\|_{L^1(\mathbb{R})} \le 1$, such that

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This is still far from the result by Kolmogorov and Titchmarsh where $(a_n)_{n\in\mathbb{Z}} \in c_0(\mathbb{Z})$!

The Fourier transform via operator (semi-)groups

Let $(U(t))_{t\in\mathbb{R}} \subset \mathcal{L}(L^2(\mathbb{R}))$ be a family of unitary operators on $L^2(\mathbb{R})$ defined by

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Observe that for fixed $f, g \in L^2(\mathbb{R})$:

$$(U(t)f, g) = \int_{\mathbb{R}} e^{-its} f(t)\overline{g(t)}\, dt = \int_{\mathbb{R}} e^{-its} \varphi(t)\, dt, \quad \varphi := f\bar{g} \in L^1(\mathbb{R}).$$

$$|(U(t)f, g)| = |(U(-t)f, g)|, \qquad t \geq 0.$$

it is enough to study bounds for one-sided weak orbits of $(U(t))_{t\in\mathbb{R}}$.

NOTE: $(U(t))_{t\in\mathbb{R}}$ is a strongly continuous operator (semi-)group.

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NOTE: The *weak orbit* $(U(t)f, g)$ of $(U(t))_{t\in\mathbb{R}}$ is the Fourier transform of the $L^1(\mathbb{R})$-function $f\bar{g}$.

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For a C_0-sem. $(T(t))_{t\geq 0}$ on a Ban. space X with generator A define

$$\begin{aligned} s(A) &:= \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} \\ s_b(A) &:= \inf\{\omega \in \mathbb{R} : R(\lambda, A) \text{ is uniformly bounded for } \mathrm{Re}\,\lambda \geq \omega\} \\ \omega_0 &:= \limsup_{t\to\infty} \frac{\ln \|T(t)\|}{t} \end{aligned}$$

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For any $a \geq b \geq c$ there exist a reflexive Banach space X and a C_0-semigroup $(T(t))_{t\geq 0}$ on X :

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If X is Hilbert, then $\omega_0 = s_b$.

What can we say about the size of weak orbits

$$\langle T(t)x, x^* \rangle, \qquad t \ge 0, x \in X, x^* \in X^*,$$

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A bit more notation: Let $A : D(A) \subset X \to X$ be a densely defined closed operator with $\rho(A) \neq \emptyset$. Define

$$C^\infty(A) := \bigcap_{n=1}^{\infty} D(A^n) \qquad \text{and note} \quad \overline{C^\infty(A)} = X.$$

Decay of weak orbits, spectral conditions

Theorem (Müller-T.)

Let $(T(t))_{t\geq 0}$ be a weakly stable C_0-semigroup a Hilbert space H with generator A ($T(t) \to 0, t \to \infty$ in WOT).

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b) Suppose that $\sigma(A) \cap i\mathbb{R} \neq \emptyset$. Then there exists $x \in C^\infty(A)$ such that $\|x\| < \sup\{f(t) : t \geq 0\} + \epsilon$ and

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Weak stability of $(T(t))_{t\geq 0}$ is essential !

For $S \subset \mathbb{R}_+$ define its *density* $d(S)$ as

$$d(S) = \lim_{t\to\infty} \frac{\mathrm{mes}(S \cap [0, t])}{t}$$

whenever the limit exists.

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Applications to function theory: back to Kolmogorov-Titchmarsh

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Recall, that by Wiener's theorem, if μ has no atoms, then

$$\widehat{\mu}(t) \to 0, \quad \text{when } t \to \infty \quad \text{along } B \text{ with } \mathrm{dens}(B) = 1.$$

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Theorem (Müller-T., '17)

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator A, such that $s_b(A) \geq 0$. Let $f : [0,\infty) \to (0,\infty)$ be a bounded function such that $\lim_{t\to\infty} f(t) = 0$. Then there exist $x \in X$, $x^* \in X^*$ and a measurable $B \subset [0,\infty)$ such that $\overline{\mathrm{Dens}}\, B = 1$ and

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Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator A, such that $s_b(A) \geq 0$. Let $f : [0,\infty) \to (0,\infty)$ be a bounded function such that $\lim_{t\to\infty} f(t) = 0$. Then there exist $x \in X$, $x^* \in X^*$ and a measurable $B \subset [0,\infty)$ such that $\overline{\mathrm{Dens}}\, B = 1$ and

$$|\langle T(t)x, x^*\rangle| \geq f(t), \qquad t \in B.$$

Lemma. Let A, B be finite sets, $|A| \geq |B| \geq 2$. Let $M \subset A \times B$, $|M| \geq |A| + |B| - 1$. Then there exist $a, a' \in A$, $b, b' \in B$ such that $a' \neq a$, $b' \neq b$ and $(a,b), (a,b'), (a',b) \in M$.

Theorem (Müller-T., '17)

Let $(T(t))_{t\geq 0}$ *be a* C_0*-semigroup on a reflexive Banach space* X *with generator* A, *such that* $s_b(A) \geq 0$. *Let* $\epsilon > 0$ *be fixed. Then*

(i) *there exist* $x \in X$, $x^* \in X^*$ *and a measurable* $B \subset [1, \infty)$ *with* $\mathrm{Dens}\, B = 1$ *such that*

$$|\langle T(t)x, x^* \rangle| \geq \frac{1}{t^{1/2+\epsilon}}, \qquad t \in B.$$

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Optimality is widely open ...