Critical points of Blaschke products
Stieltjes polynomials and moment problems

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IWOTA Chemnitz, August 2017

This talk is dedicated to the memory of

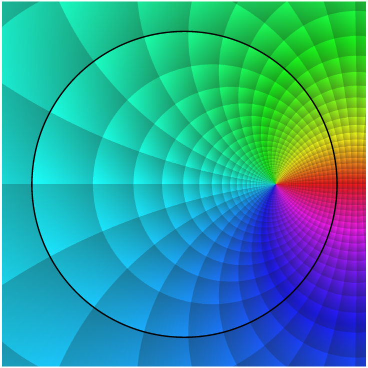
Georg Heinig

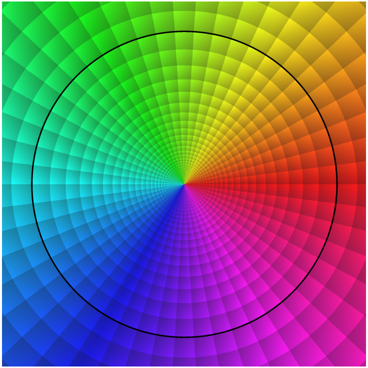
Blaschke Products

Finite Blaschke products

A finite Blaschke product of degree n is a rational function

$$B(z) = c \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a}_k z}, \qquad |a_k| < 1, \quad |c| = 1.$$





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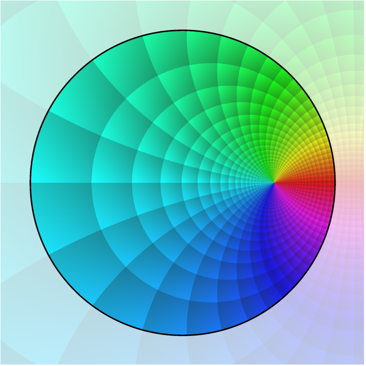

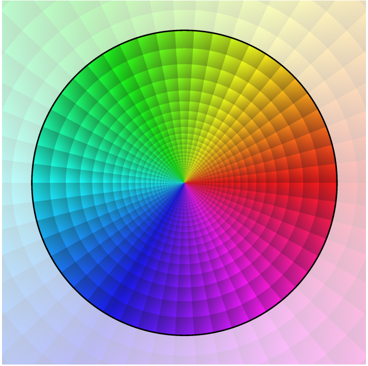

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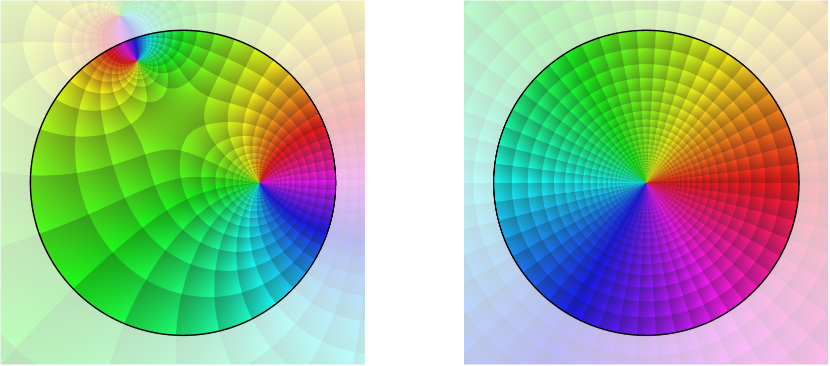

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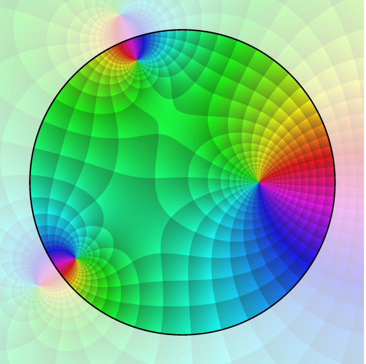

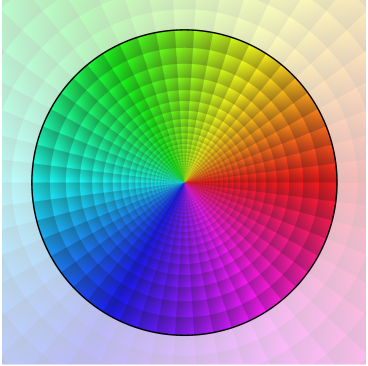

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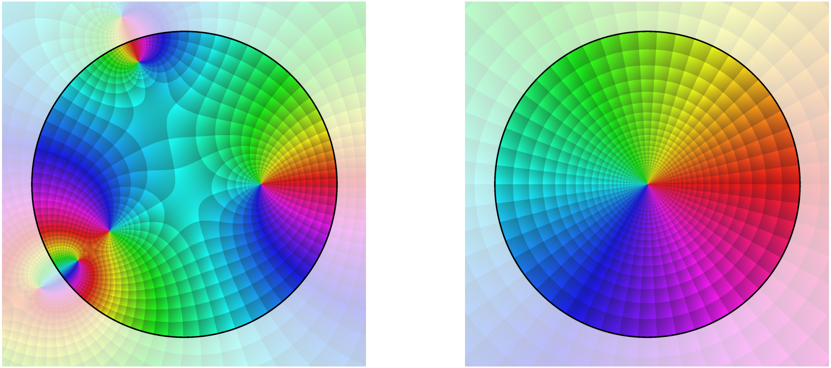

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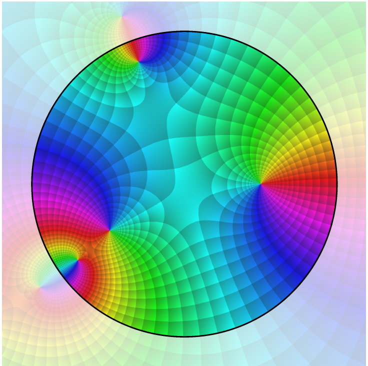

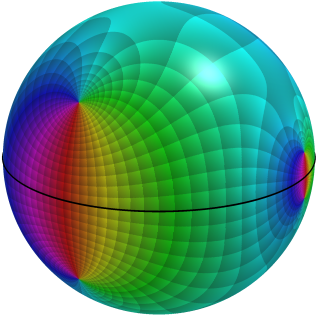

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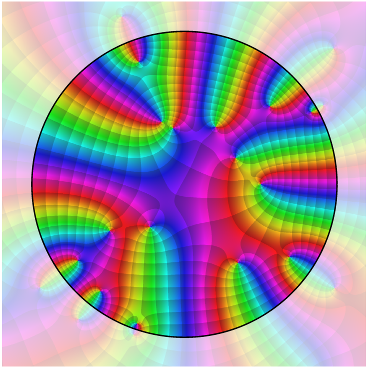

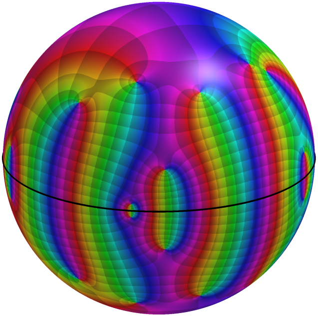

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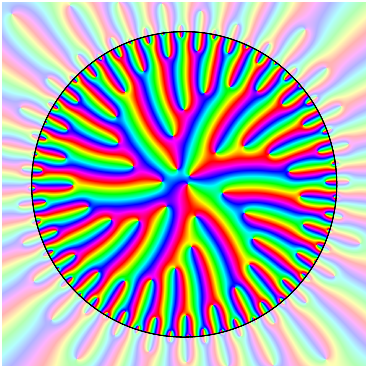

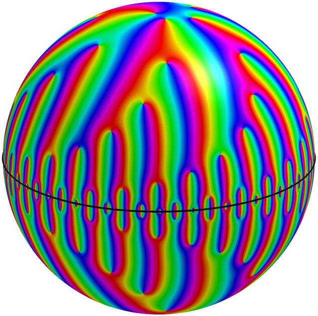

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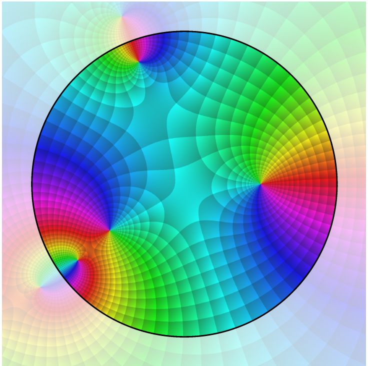
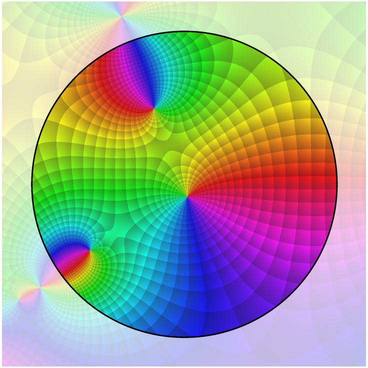
Critical points of Blaschke products

Critical points of a function are the zeros of its derivative.





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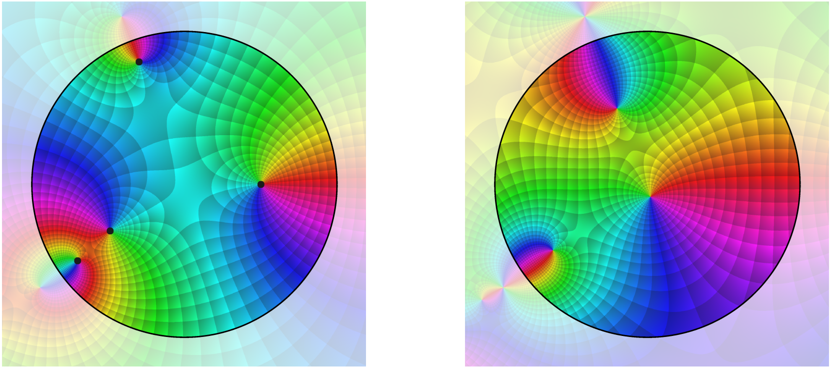

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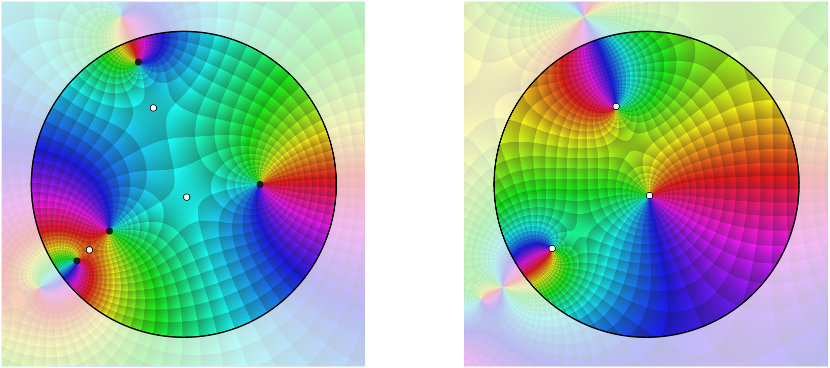

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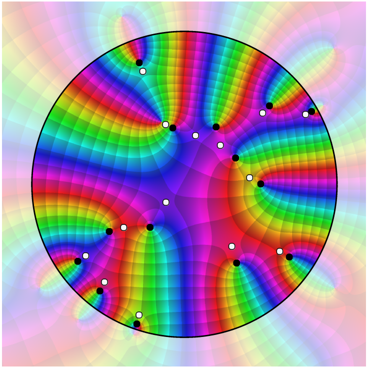

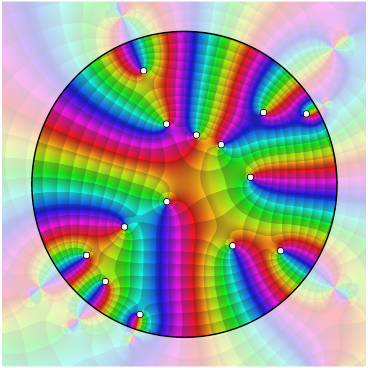

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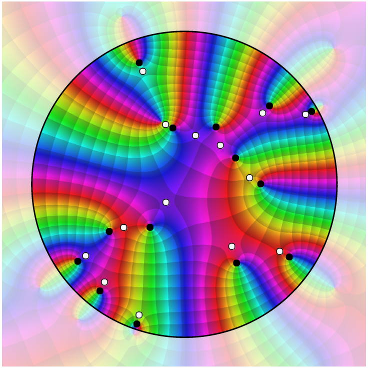

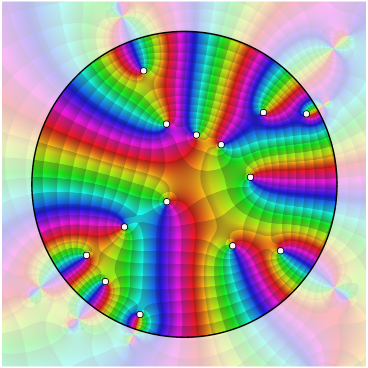

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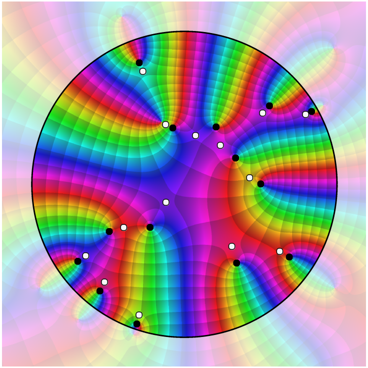

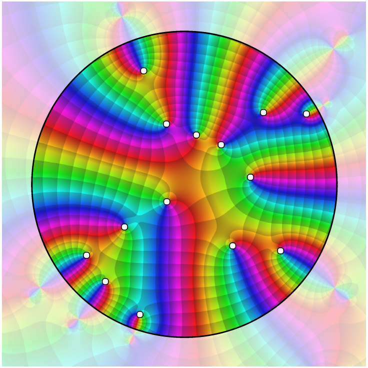

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We are interested in the interplay between zeros of B and of B'.

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Theorem

Let $\xi_1, \ldots, \xi_{n-1}$ be $n-1$ points in $\mathbb{D}$. Then there is a Blaschke product B of degree n with critical points ξ_k. B is unique up to post-composition with a conformal automorphism of $\mathbb{D}$.

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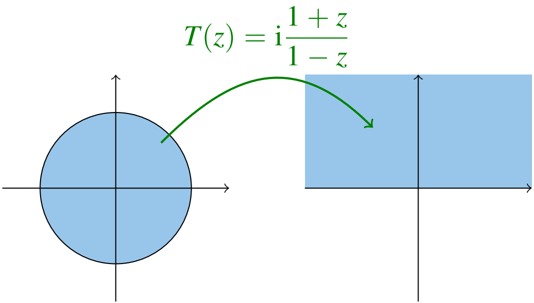
How can the zeros of B be determined from its critical points ?

Transformation to the upper half plane

The (inverse) Cayley transform sends unit disk to upper half plane



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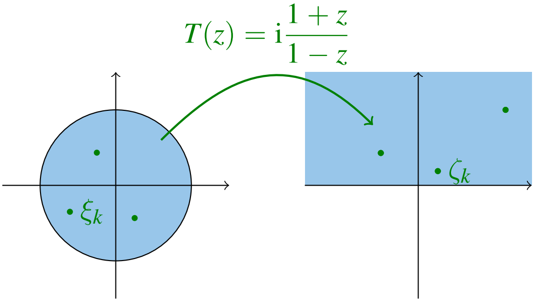

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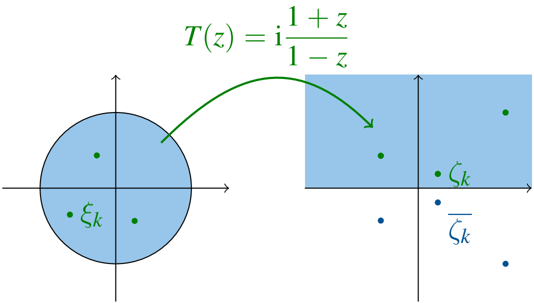

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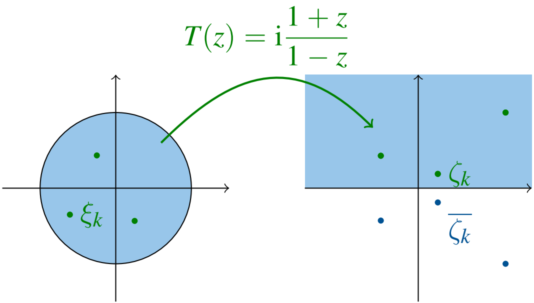

- The composition $f := T \circ B \circ T^{-1}$ is a real rational function with critical points $\zeta_k := T(\xi_k)$ and $\overline{\zeta_k}$ for $k = 1, \ldots, n-1$.
- The form of f depends on the normalization of B.

A problem for rational functions

- If $B(1) = -1$, then $f := T \circ B \circ T^{-1}$ has the form

$$f(x) = -\frac{r_1}{x - x_1} - \frac{r_2}{x - x_2} - \ldots - \frac{r_n}{x - x_n} \qquad (1)$$

with $x_1 < \ldots < x_n$ and $r_k > 0$.

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- If $B(1) = 1$, then $g := T \circ B \circ T^{-1}$ has the form

$$g(x) = ax + b - \frac{s_1}{x - t_1} - \ldots - \frac{s_{n-1}}{x - t_{n-1}} \tag{2}$$

  with $a > 0$, $b \in \mathbb{R}$, $t_1 < \ldots < t_{n-1}$, $s_k > 0$.

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Transformed problem: Find rational functions f of form (1) and g of form (2) with given critical points ζ_k in the upper half-plane $\mathbb{H}$.

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A rational function f of the form

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has critical points $\zeta_1, \ldots, \zeta_{n-1}$ if and only if

$$f'(x) = \frac{r_1}{(x - x_1)^2} + \frac{r_2}{(x - x_2)^2} + \ldots + \frac{r_n}{(x - x_n)^2} = \frac{cP(x)}{Q(x)^2} \tag{3}$$

where

$$P(x) := \prod_{k=1}^{n-1} (x - \zeta_k)(x - \overline{\zeta}_k), \qquad Q(x) := \prod_{k=1}^{n} (x - x_k) \tag{4}$$

and $c := r_1 + \ldots + r_n$.

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and $c := r_1 + \ldots + r_n$. Note that P is positive on $\mathbb{R}$.

A problem for rational functions

Condition (3) is equivalent to

$$cP(x) = \sum_{k=1}^{n} r_k \prod_{\substack{j=1 \\ j \neq k}}^{n} (x - x_j)^2 \qquad \Leftrightarrow \qquad P(x) = \sum_{k=1}^{n} P(x_k) Q_k(x)^2 \qquad (5)$$

with the *Lagrange interpolation polynomials*

$$Q_k(x) := \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{x - x_j}{x_k - x_j} = \frac{Q(x)}{Q'(x_k)(x - x_k)}, \qquad k = 1, 2, \ldots, n.$$

Here P is given and the x_k (and hence Q) have to be determined.

A problem for polynomials (first normalization)

Comparing (5)

$$P(x) = \sum_{k=1}^{n} P(x_k) Q_k(x)^2$$

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Comparing (5) with the *Lagrange-Hermite interpolation formula* for P

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First normalization: Given a positive polynomial P of degree $2n - 2$, find real polynomials R of degree $2n - 4$ such that the ODE

$$PQ'' - P'Q' + RQ = 0$$

has a polynomial solution Q of degree n with simple real roots.

Stieltjes Polynomials

Stieltjes and Van Vleck polynomials

Let $\deg A = p + 1$ and $\deg B = p$. A polynomial C of degree $p - 1$ is called Van Vleck polynomial, if the generalized Lamé equation

$$AQ'' + 2BQ' + CQ = 0$$

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has real roots $a_0 < a_1 < \ldots < a_p$ and

$$\frac{B(x)}{A(x)} = \frac{\varrho_0}{x - a_0} + \frac{\varrho_1}{x - a_1} + \ldots + \frac{\varrho_p}{x - a_p}, \quad \text{with } \varrho_k > 0.$$

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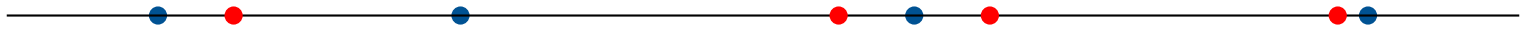
In our case: $PQ'' - P'Q' + RQ = 0$ with a positive polynomial P.

Equilibrium of Charges

Connection to problem in electrostatics

Stieltjes established a connection to electrostatics:

Put positive charges ϱ_k at the points a_k on the line. Then:



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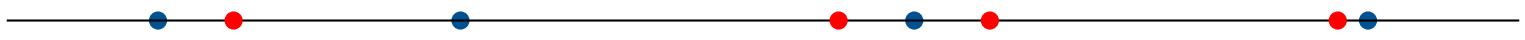
Zeros of Stieltjes polynomial Q are exactly the equilibrium points of n movable unit charges distributed between the fixed charges



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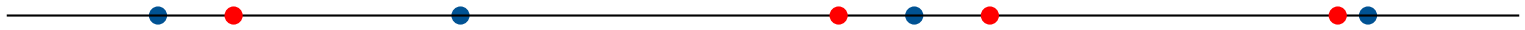

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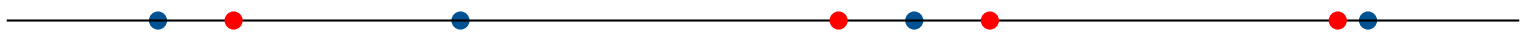
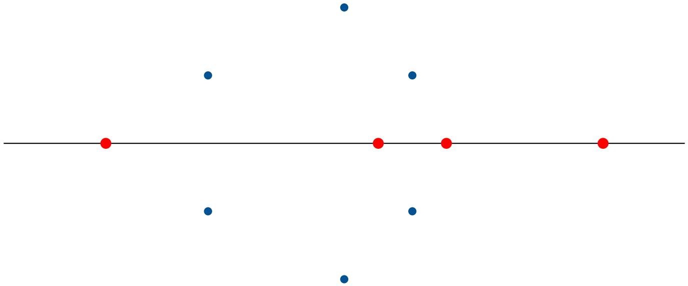


There are $\binom{n+p-1}{n}$ solutions according to the different ways to distribute the movable charges between the fixed charges.
Each solution corresponds to a minimum of the potential energy

$$W(x_1,\ldots,x_n) = -\sum_{k=0}^{p}\sum_{j=1}^{n} \varrho_k \log|a_k - x_j| - \sum_{1\le k<j\le n} \log|x_k - x_j|$$

Equilibrium problems

Zeros of Stieltjes polynomials for the ODE $PQ'' - P'Q' + RQ = 0$ are equilibrium positions of n movable unit charges on $\mathbb{R}$ in the presence of $2n-2$ charges $-\frac{1}{2}$ at the points ζ_k and $\overline{\zeta}_k$



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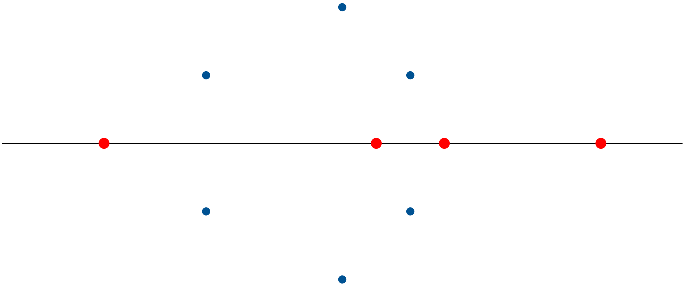

The second generalized Lamé equation $PS'' - P'S' + \tilde{R}S = 0$ leads to the same problem with $n-1$ movable unit charges.

Example: the case $n = 2$

One pair ζ and $\overline{\zeta}$ of negative charges $-\frac{1}{2}$ given.

Search equilibrium of one positive unit charge on real line.

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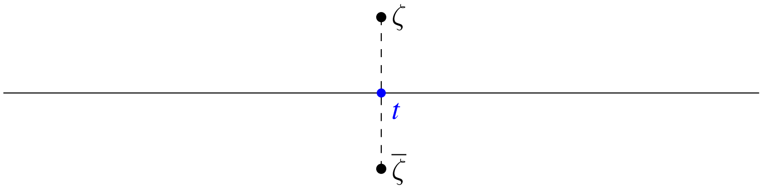

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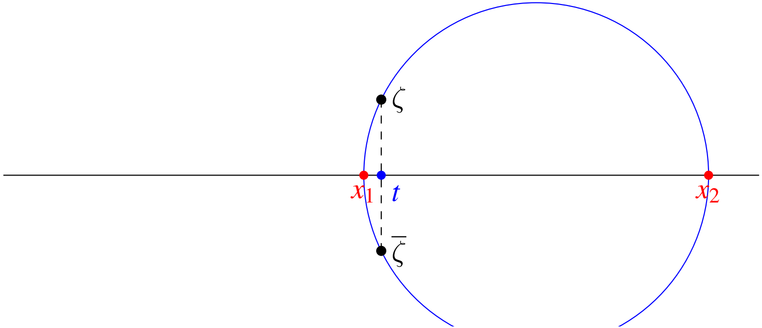

The general case

The problem with $2n-2$ fixed negative charges $-\frac{1}{2}$ and $n-1$ movable unit charges at t_j was solved by Orive and García (2010).

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Use connection with above problem: rational functions g and f differ by a Möbius transformation of upper half plane that maps 0 to ∞:

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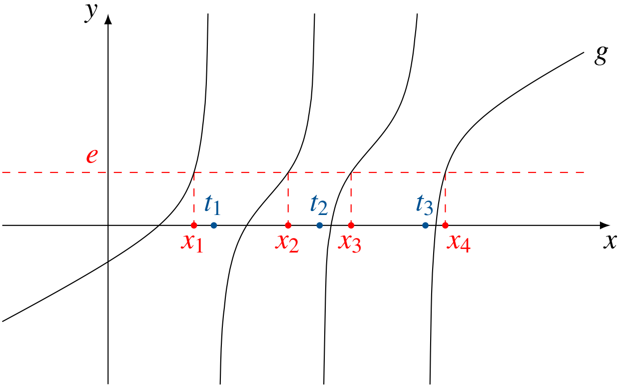
Equilibrium points of n unit charges are poles of f, i.e., solutions of $g(x)=e$ with $e\in\mathbb{R}$.

Equilibrium of n unit charges

Solutions of $g(x) = e$ with $e \in \mathbb{R}$ form one-parameter family



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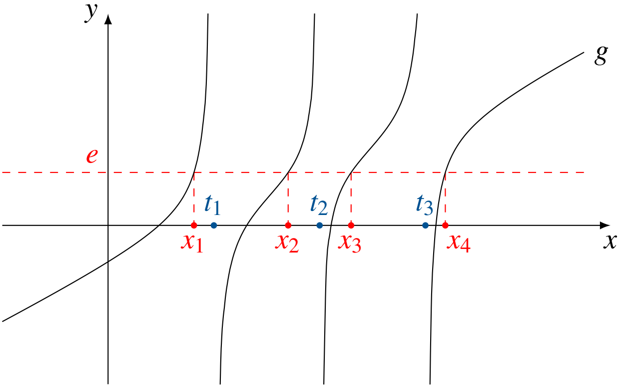

Solutions with second normalization are unique (corresponding to t_k), solutions with first normalization are not unique (corresponding to x_k).

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- the general solution of the electrostatic problem with n moveable charges (instead of $n-1$) in the case at hand,
- a description of the complete solution space of the (special) Lamé equation.

Moment Problems

Convex Cones

A *convex cone* in a real vector space is a set $\mathscr{C}$ satisfying

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Convex cone of non-negative polynomials

$$\mathscr{P}_n := \left\{(p_0, \ldots, p_{n-1}) \in \mathbb{R}^n : p_0 + p_1 x + \ldots + p_{n-1} x^{n-1} \geq 0 \text{ on } \mathbb{R}\right\},$$

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$$\mathscr{M}_n := \left\{ c = (c_0, \ldots, c_{n-1}) \in \mathbb{R}^n : c_k = \int_{-\infty}^{\infty} t^k \, d\sigma,\ \sigma \in M_n \right\},$$

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The polynomial P with $P(\zeta_j) = P(\overline{\zeta_j})$ belongs to $\mathscr{P}_{2n-1}$.

Canonical representations of moments

Any point c in the interior of the moment cone $\mathscr{M}_{2n-1}$ can be uniquely represented by an atomic measure concentrated on $n-1$ *roots* $t_1 < t_2 < \ldots < t_{n-1}$ in $\mathbb{R}$ with positive *masses* $\sigma_1, \ldots, \sigma_{n-1} > 0$ and mass $\lambda > 0$ at infinity,

$$c_k = \sum_{j=1}^{n-1} \sigma_j t_j^k, \quad (k = 0, \ldots, 2n-3), \quad c_{2n-2} = \sum_{j=1}^{n-1} \sigma_j t_j^{2n-2} + \lambda. \tag{6}$$

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There are alternative representations with masses $\varrho_j > 0$ at n roots $x_1, x_2, \ldots, x_n$, such that $x_1 < t_1 < x_2 < t_2 < \ldots < x_{n-1} < t_{n-1} < x_n$.

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Here one of the $x_j \in (t_{j-1}, t_j)$ can be fixed arbitrarily to make the representation unique.

Vandermonde decomposition of Hankel matrices

The point $c = (c_0, c_1, \ldots, c_{2n-2})$ is an inner point of the moment cone $\mathscr{M}_{2n-1}$ if and only if the Hankel matrix

$$H(c) := \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{bmatrix},$$

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is positive definite. The canonical representation involving only finite roots is equivalent to the *Vandermonde decomposition* $H(c) = VDV^\top$,

$$V := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}, \qquad D := \begin{bmatrix} \varrho_1 & 0 & \cdots & 0 \\ 0 & \varrho_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \varrho_n \end{bmatrix}.$$

We learned this from Georg Heinig and Karla Rost.

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Putting Things Together

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Is there a mapping which makes the problems isomorphic?

Yes, there is; it was introduced by Yurii Nesterov in 2000.

The Nesterov mapping

For all $c \in \mathbb{R}^{2n-1}$ with an invertible associated Hankel matrix $H(c)$, Nesterov defined the mapping

$$N : c \mapsto N(c) := H^* H(c)^{-1},$$

where H^* maps a matrix to the vector of its anti-diagonal sums

$$H^* : \mathbb{R}^{n \times n} \mapsto \mathbb{R}^{2n-1},\ X = (x_{ij}) \mapsto y = (y_k), \quad y_k := \sum_{i+j=k} x_{ij}.$$

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We gave an alternative proof using Bezoutians (inverses of Hankel matrices), which simultaneously establishes the connection between the equilibrium problem and the moment problem.

The main result for first normalization

Theorem (G. Semmler, E.W.)

Let $\zeta_1, \ldots, \zeta_{n-1}$ *be arbitrary points in the upper half plane. Denote by* $p \in \mathbb{R}^{2n-1}$ *the coefficient vector of the monic (positive) polynomial* P *of degree* $2n-2$ *with zeros* $\zeta_1, \ldots, \zeta_{n-1}$ *and* $\overline{\zeta_1}, \ldots, \overline{\zeta_{n-1}}$. *Then:*

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(ii) *The points* $x_1, \ldots, x_n$ *are equilibrium positions of* n *unit charges if and only if*

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(iii) *Let* $\xi_1, \dots, \xi_{n-1}$ *be pairwise different points in* $\mathbb{D}$*. If* $\zeta_j := T(\xi_j)$ *and*

$$f'(x) = c\,\frac{P(x)}{Q(x)^2}, \quad Q(x) := \prod_{k=1}^{n} (x - x_k),$$

then $B := T^{-1} \circ f \circ T$ *is a Blaschke product of degree* n *with critical points* $\xi_1, \dots, \xi_{n-1}$*.*

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(ii) minimization of energy functional W, e.g., gradient or Newton method. Global minimum is the one and only critical point.

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(ii) $\mathscr{E}$: equilibrium of n or $n-1$ unit charges on $\mathbb{R}$ in the presence of $2n-2$ negative charges of size $-1/2$

(iii) $\mathscr{M}$: canonical representation of a point in the interior of the moment cone

What is the best method to compute the zeros of a Blaschke product from its critical points?

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Summary and open problem

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Inversion of Nesterov mapping $N : \text{int}\, \mathscr{M}_{2d-1} \to \text{int}\, \mathscr{P}_{2d-1}$: Find a **positive definite** Bezoutian with prescribed anti-diagonal sums.

Infinite Blaschke products

Is there a direct path from problem $\mathscr{B}$ (Blaschke products) to $\mathscr{M}$ (moment problem) which avoids $\mathscr{E}$ (electrostatics)?

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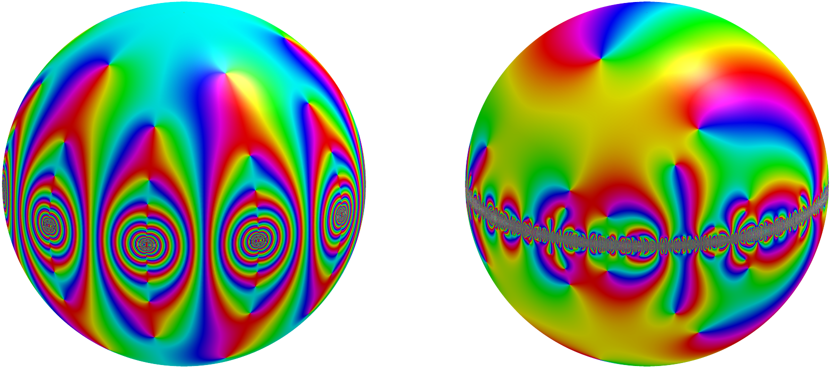
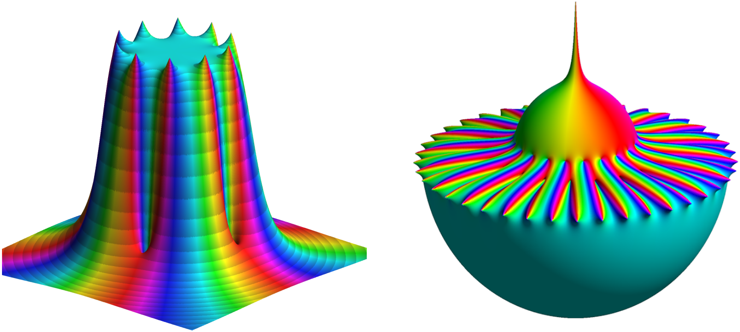


Phase plots of infinite Blaschke products on the Riemann sphere.

Software: the complex function explorer



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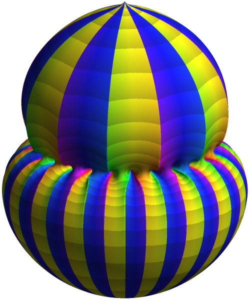

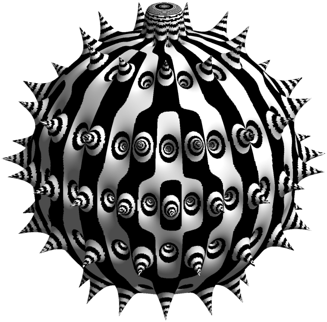

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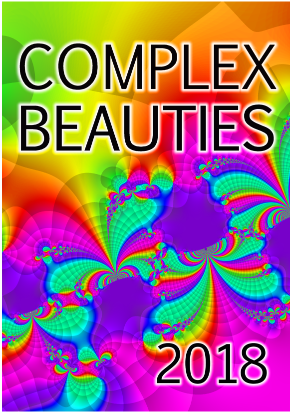

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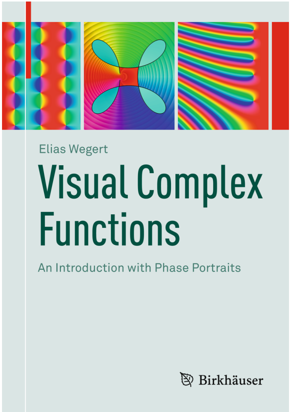

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