Lebesgue decomposition and order structure

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On the order structure of representable functionals

(to appear)

Motivation I. [Decomposition of nonnegative finite measures.]

Let $T \neq \emptyset$ be a set, $\Sigma \subseteq P(T)$ a σ-algebra, and consider the nonnegative finite **measures** μ and ν on Σ.

- ◦ $\mu \ll \nu$, if $\nu(A) = 0$ implies $\mu(A) = 0$ for all $A \in \Sigma$.
- • $\mu \perp \nu$, if $\exists P \in \Sigma : \qquad \mu(P) = \nu(T \setminus P) = 0.$

Lebesgue decomposition theorem: The measure μ splits uniquely into ν-absolute continuous and ν-singular parts:

$$\mu = \mu_{\mathrm{a}} + \mu_{\mathrm{s}}.$$

Comments:

(a) The set of measures is partially ordered by the relation

$$\mu \le \nu \qquad \Longleftrightarrow \qquad \forall A \in \Sigma : \quad \mu(A) \le \nu(A).$$

(b) This partial order is a lattice order, where the infima is

$$\big(\mu \wedge \nu\big)(A) = \inf_{P \in \Sigma} \{\mu(A \cap P) + \nu(A \setminus P)\}.$$

Consequently, we can use lattice theoretic techniques.

(c) Observe that $\mu \perp \nu$ if and only if $\mu \wedge \nu$ is the zero measure.

(d) Furthermore, $\mu \ll \nu$ if and only if $\mu = \sup\{\mu \wedge n\nu \,|\, n \in \mathbb{N}\}$.

(e) Absolute continuity is hereditary in the following sense

$$\mu \ll \nu \text{ and } \vartheta \le \mu \;\text{ imply }\; \vartheta \ll \nu.$$

(f) The decomposition is unique.

Motivation II. [Decomposition of bounded positive operators.]

Let $\mathscr{H}$ be a complex Hilbert space, and let denote $\mathrm{B}_+(\mathscr{H})$ the cone of bounded positive operators with the usual partial order

$$A \leq B \quad \Leftrightarrow \quad \forall x \in \mathscr{H}: \ (Ax \,|\, x) \leq (Bx \,|\, x).$$

For $A, B \in \mathrm{B}_+(\mathscr{H})$ we say that

- ◦ $A \ll B$, if $A = (s) \lim\limits_{n \in \mathbb{N}} A_n$ with some $(A_n)_{n \in \mathbb{N}}$ satisfying $\forall n \in \mathbb{N}: \ 0 \leq A_n \leq A_{n+1}$ and $A_n \leq c_n B$ with some $c_n \geq 0$.
- $A \perp B$, if $\operatorname{ran} A^{1/2} \cap \operatorname{ran} B^{1/2} = \{0\}$.

Ando's theorem: If A and B are bounded positive operators, then A splits into B-absolute continuous and B-singular parts

$$A = A_{\mathrm{a}} + A_{\mathrm{s}}.$$

Comments:

(a) The partially ordered set of positive operators is not a lattice.

(b) For bounded positive operators A and B it is not so easy to present a common nonzero lower bound. As we will see, the parallel sum is a good choice:

$$\big((A:B)x \,\big|\, x\big) = \inf_{y+z=x} \big\{(Ay\,|\,y) + (Bz\,|\,z)\big\} \qquad (x \in \mathscr{H}).$$

With this operation we can imitate lattice techniques.

(c) Absolute continuity is not hereditary, that is

$$A \ll B \text{ and } C \leq A \;\text{ do not imply }\; C \ll B.$$

(d) The decomposition is not unique in general.

Four **if and only if** theorems of Tsuyoshi Ando:

Let us introduce the notation $[B]A := (s) \lim_{n\to\infty} A : nB$.
This is the so called B-regular part (or generalized short) of A.

(A) $A \perp B$ **if and only if** $A : B$ is the zero operator,

(B) $A \ll B$ **if and only if** $A = [B]A$.

(C) The greatest lower bound in $\mathrm{B}_+(\mathscr{H})$ exists **if and only if**

$$[A]B \leq [B]A \qquad \text{or} \qquad [B]A \leq [A]B.$$

(D) Ando's decomposition is unique **if and only if**

$$[B]A \leq cB \quad \text{for some} \quad c \geq 0.$$

Motivation III. [Representable functionals.]

Let $\mathscr{A}$ be a complex *-algebra. A linear functional f is **representable** if there exists a *-representation $\pi : \mathscr{A} \to \mathrm{B}(\mathcal{H})$ of $\mathscr{A}$ in a Hilbert space $\mathcal{H}$ with a vector $\xi \in \mathcal{H}$ such that

$$f(a) = \big(\pi(a)\xi \,|\, \xi\big) \qquad (a \in \mathscr{A}).$$

- ◦ $f \ll g$, if $f\big((a_n - a_m)^*(a_n - a_m)\big) \to 0$ and $g(a_n^* a_n) \to 0$ imply $f(a_n^* a_n) \to 0$ for all $(a_n)_{n\in\mathbb{N}}$.
- • f and g are singular if there exists an $(a_n)_{n\in\mathbb{N}}$ in $\mathscr{A}$ such that

$$g(a_n^* a_n) \to 0 \quad \text{and} \quad f((a_n - a_m)^*(a_n - a_m)) \to 0 \quad \text{hold, and}$$

$$f(a) = \lim_{n\in\mathbb{N}} f(a_n^* a) \quad \text{for all} \quad a \in \mathscr{A}.$$

Gudder's decomposition: If $\mathcal{A}$ is a unital Banach-* algebra, then f splits into g-absolutely continuous and g-singular parts

$$f = f_{\mathrm{a}} + f_{\mathrm{s}}.$$

Questions:

From now on $\mathcal{A}$ always stands for a not necessarily unital complex $*$-algebra. The set of representable functionals (denoted by $\mathscr{A}^\sharp$) is partially ordered by the relation

$$f \leq g \qquad \Longleftrightarrow \qquad \forall a \in \mathscr{A}: \quad f(a^*a) \leq g(a^*a).$$

(q1) Given two representable functionals f and g, can we "easily" pick a nonzero representable functional h such that

$$h \leq f \quad \text{and} \quad h \leq g?$$

(q2) Does this partial order have anything to do with singularity and absolute continuity?

(q3) Is the Lebesgue (or $[\ll, \perp]$-type) decomposition unique?

(q4) Does the greatest lower bound (in $\mathscr{A}^\sharp$) of f and g exist?

(a1) Parallel sum of representable functionals:

Consider the GNS triplets $(\mathcal{H}_f, \pi_f, \xi_f)$ and $(\mathcal{H}_g, \pi_g, \xi_g)$.

Let $\pi : \mathcal{A} \to B(\mathcal{H}_f) \oplus B(\mathcal{H}_g)$ be the direct sum of π_f and π_g.

Let P be the orthogonal projection onto the following subspace

$$\{\pi_f(a)\xi_f \oplus \pi_g(a)\xi_g \,|\, a \in \mathscr{A}\}^{\perp} \subseteq \mathcal{H}_f \oplus \mathcal{H}_g.$$

Tarcsay proved that the functional $f : g$ defined by

$$(f : g)(a) := \big(\pi(a)P(\xi_f \oplus 0) \,\big|\, P(\xi_f \oplus 0)\big) \qquad (a \in \mathscr{A})$$

is representable and it satisfies

$$\big(f : g\big)(a^*a) = \inf\big\{f((a-b)^*(a-b)) + g(b^*b) \,\big|\, b \in \mathscr{A}\big\}, \qquad (a \in \mathscr{A}).$$

Recap:

$$\big(\mu \wedge \nu\big)(A) = \inf_{P \in \Sigma} \{\mu(A \cap P) + \nu(A \setminus P)\}, \qquad (A \in \Sigma)$$

$$\big((A:B)x \,\big|\, x\big) = \inf_{y \in \mathscr{H}} \big\{(A(x-y) \,|\, x-y) + (By \,|\, y)\big\}, \qquad (x \in \mathscr{H})$$

$$\big(f:g\big)(a^*a) = \inf_{b \in \mathscr{A}} \big\{f((a-b)^*(a-b)) + g(b^*b)\big\}, \qquad (a \in \mathscr{A})$$

Some properties of parallel addition:

I want to highlight only (d) and (e), because it shows that parallel addition is again a good operation to find a common lower bound.

(a) $f : g = g : f$,

(b) $(f : g) : h = f : (g : h)$,

(c) $(\lambda f) : (\lambda g) = \lambda(f : g)$,

(d) $f : g \leq f$ and $f : g \leq g$,

(e) $f_1 \leq f_2,\ g_1 \leq g_2 \quad \Longrightarrow \quad f_1 : g_1 \leq f_2 : g_2$,

(f) $f_n \downarrow f \quad \Longrightarrow \quad f_n : g \downarrow f : g$,

(g) $(f_1 : g_1) + (f_2 : g_2) \leq (f_1 + f_2) : (g_1 + g_2)$,

(h) $(\alpha f) : (\beta f) = \frac{\alpha\beta}{\alpha+\beta} f$.

(a2) Absolute continuity and singularity

In full analogy with the bounded positive operator case, we can define the regular part of f with respect to g, that is

$$[g]f := \sup_{n\in\mathbb{N}} f : ng.$$

Furthermore, we can characterize $\ll$ and $\perp$ as follows:

$$f \ll g \quad\Longleftrightarrow\quad [g]f = f \qquad \text{and} \qquad f \perp g \quad\Longleftrightarrow f : g = 0.$$

In fact, we can prove that both absolute continuity and singularity can be formulated by means of the partial order.

- $f \ll g$ if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{A}$ such that

$$f_n \leq c_n g \qquad \text{and} \qquad f = \sup_{n\in\mathbb{N}} f_n.$$

- $f \perp g$ if $h \leq f$ and $h \leq g$ imply that $h = 0$ for all $h \in \mathscr{A}^\sharp$.

(a3) Lebesgue decomposition of representable functionals

Now, putting all these things together, we can prove the following theorem by elementary algebraic manipulation.

Theorem: Let $\mathscr{A}$ be a complex $*$-algebra. Let $f, g \in \mathscr{A}^\sharp$ be arbitrary representable functionals on $\mathscr{A}$. Then

$$f = [g]f + (f - [g]f)$$

is a Lebesgue decomposition of f with respect to g. That is, $[g]f \ll g$ and $(f - [g]f) \perp g$. Furthermore, this decomposition is extremal in the following sense:

$$h \in \mathscr{A}^\sharp,\ h \leq f \text{ and } h \ll g \quad \Rightarrow \quad h \leq [g]f.$$

If $[g]f \leq c \cdot g$ for some $c \geq 0$, then the decomposition is unique.

Examples

The following two examples are devoted to demonstrate that the sufficient condition can be redundant, and also can be necessary.

E1. If the algebra $\mathscr{A}$ is finite dimensional, then the Lebesgue decomposition is unique for all $f, g \in \mathscr{A}^\sharp$.

E2. Let $\mathscr{A}$ be the Hilbert-algebra of Hilbert-Schmidt operators. Now a functional f is representable if and only if it is of the form

$$f(A) = \operatorname{Tr}(FA) \qquad (A \in \mathscr{A})$$

with a suitable positive trace class operator F. Combining the properties of the mapping $f \mapsto F$ with Ando's characterization, one can prove that our condition is necessary and sufficient.

(a4) The infimum problem in $\mathscr{A}^\sharp$

We say that the infimum of two representable functionals f and g exists in $\mathscr{A}^\sharp$ if there is a common lower bound $h \in \mathscr{A}^\sharp$ which is greater then any other common lower bound $h' \in \mathscr{A}^\sharp$. That is,

$$h' \in \mathscr{A}^\sharp; \qquad h' \leq f \quad \text{and} \quad h' \leq g \quad \Longrightarrow \quad h' \leq h.$$

The infimum of f and g (in case if it exists) is denoted by $f \wedge g$.

Theorem: Let f and g be representable functionals on the not necessarily unital $*$-algebra $\mathscr{A}$. If $[f]g$ and $[g]f$ are comparable,

$$\text{that is, either} \quad [f]g \leq [g]f \quad \text{or} \quad [g]f \leq [f]g,$$

then the infimum $f \wedge g$ exists in $\mathscr{A}^\sharp$. In this case,

$$f \wedge g = \min\{[f]g, [g]f\}.$$

Examples

The following two examples will show that our sufficient condition can be redundant, and also can be necessary.

E3. Let $\mathscr{A}$ be a unital commutative C^*-algebra. Recall that every representable functional f on $\mathscr{A}$ can be identified as a non-negative finite regular Borel measure μ_f over the maximal ideal space of $\mathscr{A}$. Using this $f \mapsto \mu_f$ identification one can prove that the infimum of any two functionals exists.

E4. Let $\mathscr{A}$ be the C^*-algebra of all compact operators on a fixed Hilbert space $\mathcal{H}$. Then every representable functional f can be identified with a trace class operator F satisfying

$$f(A) = Tr(FA) \qquad (A \in \mathscr{A})$$

Again, combining the properties of this correspondence with Ando's characterization, one can prove that the infimum of two representable functionals exists if and only if their corresponding regular parts $[f]g$ and $[g]f$ are comparable.

Extreme points of intervals

Closing this talk, we are going to describe the extreme points of convex sets (or intervals) of the form

$$[0, f] := \big\{h \in \mathscr{A}^\sharp \,\big|\, 0 \leq h \leq f\big\},$$

where $f \in \mathscr{A}^\sharp$ is fixed.

Theorem: Let f be a representable functional on $*$-algebra $\mathscr{A}$. Then the following statements are equivalent for $g \in \mathscr{A}^\sharp$:

(i) g is an extreme point of $[0, f]$,

(ii) $g : (f - g) = 0$,

(iii) $[g]f = g$.

Finally, we mention that the partially ordered set $\Big(\mathrm{ex}[0, f], \leq\Big)$ is a lattice. $\Big[$Namely, $g_1 \curlywedge g_2 = 2(g_1 : g_2)$ and $g_1 \curlyvee g_2 = [g_1 + g_2]f.\Big]$

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Thank you for your attention!