Spectral Asymptotics for Helson Matrices

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$$H(b) = \{b(j+k)\}_{j,k\geq 0}, \quad \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+).$$

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(H^2 is the usual Hardy space on the unit disk).

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Important realisation as operators on $\mathcal{H}^2$
($\mathcal{H}^2$ is the Hardy space of Dirichlet series on half plane $\{\mathrm{Re}\, s > 1/2\}$).

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- Spectral properties of $M(a)$ and $H(b)$ coincide (Brevig, Perfekt, Seip, Siskakis, Vukotić, Perfekt, Pushnitski).

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Theorem (Widom 1966)

For $\gamma > 1$, let $b(j) = 1/(1+j)^\gamma$, $j \geq 0$. Then $H(b) \geq 0$ and

$$\lambda_n^+(H(b)) = \exp\left(-\pi\sqrt{2\gamma n} + o(\sqrt{n})\right), \quad n \to \infty.$$

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Theorem (Pushnitski – Yafaev 2015)

For $\alpha > 0$, let $b(j) = 1/j(\log j)^\alpha$ for all sufficiently large j. Then $H(b)$ is compact and

$$\lambda_n^+(H(b)) = C(\alpha)n^{-\alpha} + o(n^{-\alpha}), \quad \lambda_n^-(H(b)) = O(n^{-\alpha-1}), \quad n \to \infty$$

where

$$C(\alpha) = 2^{-\alpha}\pi^{1-2\alpha}B(\tfrac{1}{2\alpha}, \tfrac{1}{2})^\alpha.$$

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- Not known whether an analogue of Widom's Theorem holds for Helson matrices.

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$$a(j) = \underbrace{\int_0^c |\log j|^{-\alpha} j^{-\frac{1}{2}-\lambda} d\lambda}_{\widetilde{a}(j)} \quad + \quad \text{error}$$

where $0 < c < 1$ and

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The result follows by application of standard spectral perturbation theory.

Integral Hankel and Helson operators

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Hankel integral operator: for a complex valued function $\mathbf{b}$ on $\mathbb{R}_+$, $\mathbf{H}(\mathbf{b}) : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$,

$$\mathbf{H}(\mathbf{b})f : x \mapsto \int_0^\infty \mathbf{b}(x+y)f(y)dy, \quad x > 0.$$

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Helson integral operator: for a complex valued function $\mathbf{a}$ on $(1, \infty)$,
$\mathsf{M}(\mathbf{a}) : L^2(1, \infty) \to L^2(1, \infty)$,

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(i) *If* $\mathbf{a}(t) = \int_0^c |\log t|^{-\alpha} t^{-\frac{1}{2}-\lambda} d\lambda$, *then* $\mathbf{M}(\mathbf{a}) \geq 0$ *and*

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(ii) *If* $\mathbf{a}(t) = O(t^{-1/2}(\log t)^{-1}(\log\log t)^{-\alpha-1}), \quad t \to \infty,$ *then*

- $\mathbf{M}(\mathbf{a}) \in \mathbf{S}_p$ *for any* $p > 1/(\alpha+1)$;
- *and* $\mathbf{M}(\mathbf{a}) \in \mathbf{S}_{p,\infty}$ *with* $p = 1/(\alpha+1)$.

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Theorem (M – Pushnitski 2017)

Let $w \geq 0$ be a bounded function on $\mathbb{R}_+$ with bounded support. Let

$$\mathbf{a}(t) = \int_0^\infty t^{-\frac{1}{2}-\lambda} w(\lambda) d\lambda, \quad t > 1, \quad a := \mathbf{a}|_{\mathbb{N}}.$$

Then $M(a) \approx \mathbf{M}(\mathbf{a}) + A$ with $A \in \cap_{p>0} \mathbf{S}_p$.

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- In particular, $M(a)$ and $\mathbf{M}(\mathbf{a})$ obey the same spectral asymptotics.

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$$\mathcal{N} : L^2(\mathbb{R}_+) \to \ell^2(\mathbb{N}), \quad f \mapsto \left\{ \int_0^\infty j^{-x-\frac{1}{2}} w(x)^{1/2} f(x)\,dx \right\}_{j=1}^{\infty}.$$

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Use aymptotics of ζ-function and properties of the Laplace transform to write

$$w^{1/2}\{\mathbf{H}(\zeta(x+1)) - \mathbf{H}(1/x)\}w^{1/2} = \mathbf{H}(\mathbf{b}_1) + U^*\mathbf{H}(\mathbf{b}_2)U + \text{rank one}$$

where $\mathbf{b}_1$ and $\mathbf{b}_2$ are in the Schwartz class and U is unitary.

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Assume that $\mathbf{M}(\mathbf{a})$ *is bounded and belongs to the Schatten class* $\mathbf{S}_p$ *with* $0 < p \leq 1$. *Set* $a(j) = \mathbf{a}(j)$, $j \geq 2$, *and* $a(1) = 0$. *Then we have the norm bounds*

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- Note that $\mathbf{M}(\mathbf{a}) \in \mathbf{S}_1 \implies \mathbf{a}(t)$ is continuous in $t > 1$.

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$$\mathbf{a} = \sum_{n\in\mathbb{Z}} \mathbf{a}_n, \quad \mathrm{supp}(\mathbf{a}_n) \subseteq \left[\exp\left(2^{n-1}\right), \exp\left(2^{n+1}\right)\right].$$

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Use the Plancherel-Polya inequality to estimate

$$||M(a_n)||_{\mathbf{S}_p}^p \leq C_p 2^n \int_0^\infty \left| \int_{-\infty}^\infty \frac{a_n(t)}{\sqrt{t}} t^{is} \, dt \right|^p ds.$$

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Then use Peller's characterisation of Hankel operators of class $\mathbf{S}_p$ to get

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For part (ii), use (i) and real interpolation between Besov spaces.

Thank you!