Let X be a (reduced) complex space, and let $A, B : X \to \mathrm{Mat}(n \times n, \mathbb{C})$ be holomorphic.

Definition. A and B are called (globally) holomorphically similar on X if $\exists$ holomorphic $H : X \to \mathrm{GL}(n, \mathbb{C})$ s.t. $H^{-1}AH = B$ on X.

A and B are called locally holomorphically similar at $z_0 \in X$ if $\exists$ neighborhood U of z_0 s.t. $A|_U$ and $B|_U$ are holomorphically similar on U.

Correspondingly we define, for $k = 0, 1, 2, \ldots, \infty$,

(globally) $\mathcal{C}^k$ similar on X

locally $\mathcal{C}^k$ similar at a point

It seems, the first who studied the similarity of holomorphic matrices was *Wolfgang Wasow* [J. Math. Anal. Appl. **4**, 202-206 (1962)].
He first observes that "pointwise similarity" is not sufficient, for example, let

$$A(z) := \begin{pmatrix} 0 & z^2 \\ 0 & 0 \end{pmatrix}, \quad B(z) := \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}, \quad z \in \mathbb{C}.$$

Then $A(z)$ and $B(z)$ are similar for each $z \in \mathbb{C}$, but A and B are not locally $\mathcal{C}^0$ similar at 0,

Then Wasow proves the following sufficient criterion:
Consider the holomorphic map (Wasow map)

$$W_{A,B} : X \longrightarrow \mathrm{End}\,\mathrm{Mat}(n \times n, \mathbb{C})$$
$$\mathrm{Mat}(n \times n, \mathbb{C}) \ni \Phi \longmapsto A(z)\Phi - \Phi B(z), \qquad z \in X.$$

Assume that $\dim \mathrm{Ker}\, W_{A,B}(z)$ does not depend on z (Wasow condition), and that, for each $z \in X$, $A(z)$ and $B(z)$ are similar. Then A and B are locally holomorphically similar at each point of X.

Criteria for local holmorphic similarity

In general, the Wasow condition is not satisfied, and it is not necessary for local holomorphic similarity. The following (obviously necessay) criteria are sufficient:

(a) X a domain in $\mathbb{C}$, A and B locally $\mathcal{C}^0$ similar at $z_0 \in X$ $\Longrightarrow$ A and B locally holomorphically similar at z_0.

(b) X an arbitrary complex space, A and B locally $\mathcal{C}^\infty$ similar at $z_0 \in X \Longrightarrow A$ and B locally holomorphically similar at z_0.

Proof of (a): Let $N = n^2$ and let $W : X \to \mathrm{Mat}(N \times N, \mathbb{C})$ be a representation matrix of the Wasow map $W_{A,B}$. Then, for each $z_0 \in X$, in a neighborhood U of z_0, we have a Smith factorization:

$$W(z) = E(z) \begin{pmatrix} \Delta(z) & 0 \\ 0 & 0 \end{pmatrix} F(z), \quad z \in U,$$

where $E, F : U \to \mathrm{GL}(N, \mathbb{C})$ are invertible, and $\Delta(z)$ is the $N \times N$ diagonal matrix with the diagonal

$$(z - z_0)^{\kappa_1}, \ldots, (z - z_0)^{\kappa_m}, 0, \ldots, 0\ ,$$

where $\kappa_1, \ldots, \kappa_m \geq 0$ are some integers. This shows that the family $\{\mathrm{Ker}\, W(z)\}_{z \in U \setminus \{z_0\}}$, is a holomorphic sub-vector bundle of the product bundle $(U \setminus \{z_0\}) \times \mathbb{C}^N$, which extends as a holomorphic vector bundle to z_0.

(b) is more difficult and due to [K. Spallek, Math. Ann. 177, 1967, Satz 5.4, applied to the Wasow map].

Counterexample (arXiv:1703.09530)

In (b), $\mathcal{C}^\infty$ cannot be replaced with $\mathcal{C}^k$, $k < \infty$. For example, let $A, B : \mathbb{C}^2 \to \mathrm{Mat}(n \times n, \mathbb{C})$ be defined by

$$A(z,w) = \begin{pmatrix} z^{2+k}w^{2+k} & z^{3+k} \\ w^{3+k} & 0 \end{pmatrix}, \quad B(z,w) = \begin{pmatrix} 0 & z^{3+k} \\ w^{3+k} & z^{2+k}w^{2+k} \end{pmatrix}.$$

Then, one can prove

(i) A and B are locally $\mathcal{C}^k$ similar, but not locally holomorphically similar at 0;

(ii) moreover, $\exists$ 1-dim. analytic subsets X of $\mathbb{C}^2$ with $0 \in X$ s.t. $A|_X$ and $B|_X$ are not locally holomorphically similar at 0. For example,

1) $X = \{z^p = w^q\}$, where p, q are relatively prime and $k + 2 < q < p$,

2) X is the union of $2k + 5$ pairwise different 1-dimensional linear subspaces of $\mathbb{C}^2$.

Global similarity

Theorem 1. (arXiv:1703.09524, arXiv:1703.09530) Let X be a one-dimensional Stein space, and let $A, B : X \to \mathrm{Mat}(n \times n, \mathbb{C})$ be two holomorphic maps, which are locally holomorphically similar at each point of X. Then A and B are globally holomorphically similar on X.

If X is smooth (i.e., a non-compact connected Riemann surface), this was proved by R. Guralnick [Lin. Alg. Appl. 99, 85-96 (1988)]. Actually, Guralnick proves a more general algebraic theorem for matrices with elements in certain Bezout rings, and then applies this to the ring $\mathcal{O}(X)$.
This does not work if X is not smooth, or smooth an higher dimensional.
In arXiv:1703.09530 we give a proof, using Guralnick's result, passing to the normalization of X (which is smooth).
In arXiv:1703.09524 we give a proof which is independent of Guralnick's work. This proof is longer but has the advantage that it applies also to the higher dimensional case.

Theorem 2. Let X be a 2-dimensional contractible Stein manifold. Then any two holomorphic maps $A, B : X \to \mathrm{Mat}(n \times n, \mathbb{C})$, which are locally holomorphic similar at each point of X, are globally holomorphically similar on X.

Counterexample

$$A(z) := \begin{pmatrix} \left(z_1^2 + z_2^2 - 2\right)^2 + (z_1^2 + z_2^2) z_3^2 & 1 \\ 0 & 0 \end{pmatrix}, \quad z \in \mathbb{C}^3.$$

Then there exists a convex domain $X \subseteq \mathbb{C}^3$ and a holomorphic map $B : X \to \mathrm{Mat}(2 \times 2, \mathbb{C})$ which is locally holomorphically similar to A at each point of X, but not globally holomorphically similar to A on X.

To prove Theorems 1 and 2, we use the Oka principle for Oka pairs of Forster and Ramspott [Invent. mat. 1, 1966, Satz 1]. Together with Spallek's result [Math. Ann. 177, 1967, Satz 5.4], this also gives the following

Theorem 3. (arXiv:1703.09530) Let X be a Stein space (of arbitrary dimension), and let $A, B : X \to \mathrm{Mat}(n \times n, \mathbb{C})$ two holomorphic maps, which are globally $\mathcal{C}^\infty$ similar on X.

Then A and B are globally holomorphically similar on X.

Counterexample (arXiv:1703.09530)

In Theorem 3, "$\mathcal{C}^\infty$" cannot be replaced by "$\mathcal{C}^k$ with $k < \infty$" + "loc. hol. similar at each point":

There is a Stein domain X in $\mathbb{C}^5$ such that, $\forall\ k \in \mathbb{N}$, $\exists$ holomorphic maps $A, B : X \to \mathrm{Mat}(2 \times 2, \mathbb{C})$ s.t. A and B are

locally holomorphically similar at each point of X,
globally $\mathcal{C}^k$ similar on X,
but not globally holomorphically similar on X.

Question. $\mathbb{C}^4$ instead of $\mathbb{C}^5$?

On the proofs

For $\Phi \in \mathrm{Mat}(n \times n, \mathbb{C})$, we define

$$\mathrm{Com}\,\Phi = \Big\{\Theta \in \mathrm{Mat}(n \times n, \mathbb{C}) : \Phi\Theta = \Theta\Phi\Big\},$$
$$\mathrm{GCom}\,\Phi = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{Com}\,\Phi$$

Now let $A : X \to \mathrm{Mat}(n \times n, \mathbb{C})$ be a holomorphic map. Then we introduce the "bundles"

$$\mathrm{Com}\,A := \Big\{\mathrm{Com}\,A(z)\Big\}_{z \in X} \quad \Big(\subseteq X \times \mathrm{Mat}(n \times n, \mathbb{C})\Big)$$

and

$$\mathrm{GCom}\,A := \Big\{\mathrm{GCom}\,A(z)\Big\}_{z \in X} \quad \Big(\subseteq X \times \mathrm{GL}(n, \mathbb{C})\Big).$$

These "bundles" are not locally trivial, but nevertheless we have the sheaves $\mathcal{O}^{\mathrm{Com}\,A}$ and $\mathcal{O}^{\mathrm{GCom}\,A}$ of holomorphic sections, and the sheaves $\mathcal{C}^{\mathrm{Com}\,A}$ and $\mathcal{C}^{\mathrm{GCom}\,A}$ of continuous sections.

Moreover, we introduce the Forster-Ramspott sheaf $\widehat{\mathcal{O}}^{\mathrm{GCom}\,A}$, which is defined as follows: For $\emptyset \neq U \subseteq X$ open, $\widehat{\mathcal{O}}^{\mathrm{GCom}\,A}(U)$ is the group of all $f \in \mathcal{C}^{\mathrm{GCom}\,A}(U)$ s.t.,
$\forall\ \xi \in U$, $\exists$ a neigh. $V \subseteq U$ of ξ and $h \in \mathcal{O}^{\mathrm{GCom}\,A}(V)$ s.t.

$$h(\xi) = f(\xi).$$

Now let $B : X \to \mathrm{Mat}(n \times n, \mathbb{C})$ be a second holomorphic map which is locally holomorphically similar to A at each point of X, i.e., $\exists$ a covering $\{U_i\}$ of X and holom. $H_i : U_i \to \mathrm{GL}(n, \mathbb{C})$ s.t.

$$B = H_i^{-1} A H_i \quad \text{on} \quad U_i. \tag{1}$$

Then $H_i^{-1} A H_i = B = H_j^{-1} A H_j$ on $U_i \cap U_j$, and, hence,

$$A H_i H_j^{-1} = H_i H_j^{-1} A \quad \text{on} \quad U_i \cap U_j,$$

i.e.,

$$H_i H_j^{-1} \in \mathcal{O}^{\mathrm{GCom}\,A}(U_i \cap U_j)$$

is a cocycle (Cousin problem) in $\mathcal{O}^{\mathrm{GCom}\,A}$.

If this cocycle splits, i.e., if $H_i H_j^{-1} = h_i h_j^{-1}$ on $U_i \cap U_j$ for some $h_i \in \mathcal{O}^{\mathrm{GCom}\, A}(U_i)$, then

$$h_i^{-1} H_i = h_j^{-1} H_j \quad \text{on} \quad U_i \cap U_j,$$

and, hence, there is a well-defined global holomorphic map $H : X \to \mathrm{GL}(n, \mathbb{C})$ s.t. $H := H_i^{-1} h_i$ on U_i, and which satisfies

$$H^{-1} B H = h_i^{-1} H_i B H_i^{-1} h_i \overset{(1)}{=} h_i^{-1} A h_i = A,$$

i.e., B is globally holomorphically similar to A. One can prove also the opposite, so that we have the following

Statement. Each holomorphic $B : X \to \mathrm{Mat}(n \times n, \mathbb{C})$ which is locally holomorphically similar to A gives rise to a cocycle in $\mathcal{O}^{\mathrm{GCom}\, A}$, and B is globally holomorphically similar to A if and only if this cocycle splits.

It is not difficult to see that the pair $\left(\mathcal{O}^{\mathrm{GCom}\,A}, \widehat{\mathcal{O}}^{\mathrm{GCom}\,A}\right)$ is an Oka pair in the sense of [*O. Forster* and *K. J. Ramspott*, Invent. mat. 1, 1966]. Therefore the following is a special case of Satz 1 in this paper of Forster and Ramspott:

If X be a Stein space, then an $\mathcal{O}^{\mathrm{GCom}\,A}$-cocycle splits if and only if it splits as an $\widehat{\mathcal{O}}^{\mathrm{GCom}\,A}$-cocycle.

Summary. If X is a Stein space, then each holomorphic $B : X \to \mathrm{Mat}(n \times n, \mathbb{C})$ which is locally holomorphically similar to A gives rise to an $\mathcal{O}^{\mathrm{GCom}\,A}$-cocycle, and B is globally holomorphically similar to A if and only if this cocycle splits as an $\widehat{\mathcal{O}}^{\mathrm{GCom}\,A}$-cocycle.

Corollary. To prove Theorems 1 and 2, now it is sufficient to prove the following topological result:

If X is a 1-dimensional Stein space, or a contractible 2-dimensional Stein manifold, then, for each holomorphic $A : X \to \mathrm{Mat}(n \times n, \mathbb{C})$,

$$H^1(X, \widehat{\mathcal{O}}^{\mathrm{GCom}\,A}) = 0. \tag{2}$$

To prove (2), the difficulty is that $\mathrm{GCom}\, A$ is not locally trivial. For example, let

$$A(z) := \begin{pmatrix} z & 1 \\ 0 & 0 \end{pmatrix}, \quad z \in \mathbb{C}.$$

Then $A(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and, therefore,

$$\mathrm{GCom}\, A\,(0) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a \in \mathbb{C}^*, b \in \mathbb{C} \right\}.$$

On the other hand, if $z \neq 0$, then the Jordan form of $A(z)$ is $\begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}$, which shows that, for some $T(z) \in \mathrm{Gl}(n, \mathbb{C})$,

$$\mathrm{GCom}\, A\,(z) = T(z)^{-1} \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \;\middle|\; a, d \in \mathbb{C}^* \right\} T(z)$$

Hence $\pi_1\Big(\mathrm{GCom}\, A\,(0)\Big) = \mathbb{Z}$, whereas $\pi_1\Big(\mathrm{GCom}\, A\,(z)\Big) = \mathbb{Z}^2$ if $z \neq 0$.

This shows: The fiber of $\mathrm{GCom\,A}$ over 0 is even not homeomorphic to the fibers over z with $z \neq 0$ (although the bundle $\mathrm{Com}\, A$ is trivial as a holomorphic vector bundle).
The following fact helps to overcome this difficulty.

If X is a complex space and $A : X \to \mathrm{Mat}(n \times n, \mathbb{C})$ is a holomorphic map, then the set of points of X where the "Jordan structure of A changes" is an analytic subset of X, Y, which of codimension ≥ 1 everywhere in X.

This fact can be found in a book of H. Baumgärtel [Birkhäuser, 1985]. I have another proof in arXiv:1703.09535.

This is helpful, because one can prove that, over $X \setminus Y$, $\mathrm{GCom}\, A$ is locally trivial in the following sense:

For each contractible open set $W \subseteq X \setminus Y$ and each $z_0 \in W$, there exists a holomorphic map $H : W \to \mathrm{GL}(n, \mathbb{C})$ such that

$$H(z)^{-1}\mathrm{GCom}\, A(z)H(z) = \mathrm{GCom}\, A(z_0) \quad \text{for all} \quad z \in W.$$