Hankel and Helson matrices

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Introduction: Hankel and Helson

Hankel matrices: for a sequence $a(n)$, $n \in \mathbb{Z}_+$,

$$H(a) = \begin{pmatrix} a(0) & a(1) & a(2) & \dots \\ a(1) & a(2) & a(3) & \dots \\ a(2) & a(3) & a(4) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \{a(n+m)\}_{n,m\geq 0} \quad \text{in} \quad \ell^2(\mathbb{Z}_+).$$

Helson matrices: for a sequence $b(n)$, $n \in \mathbb{N}$,

$$M(b) = \begin{pmatrix} b(1) & b(2) & b(3) & \dots \\ b(2) & b(4) & b(6) & \dots \\ b(3) & b(6) & b(9) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \{b(nm)\}_{n,m\geq 1} \quad \text{in} \quad \ell^2(\mathbb{N}).$$

Introduced by H. Helson (2006) in the context of Dirichlet series

Comparison: $H(a) \leftrightarrow M(b)$

($\exists$ also multiplicative analogues of Toeplitz: $\{b(n/m)\}_{n,m\geq 1}$)

Hankel and Helson: a very naive comment

$$H(a) = \{a(n+m)\}_{n,m\geq 0}, \qquad M(b) = \{b(nm)\}_{n,m\geq 1}$$

Consider continuous analogues

Integral Hankel operators:

$$\mathbf{H}(\mathbf{a}) : f \mapsto \int_0^\infty \mathbf{a}(x+y)f(y)dy \quad \text{in } L^2(\mathbb{R}_+).$$

Integral Helson operators:

$$\mathbf{M}(\mathbf{b}) : f \mapsto \int_1^\infty \mathbf{b}(ts)f(s)ds \quad \text{in } L^2(1,\infty).$$

By change of variable $t = e^x$, $\mathbf{M}(\mathbf{b})$ is unitarily equivalent to $\mathbf{H}(\mathbf{a})$ with $\mathbf{a}(x) = e^{x/2}\mathbf{b}(e^x)$.

No such change of variable exists between $\mathbb{N}$ and $\mathbb{Z}_+$!

$\Rightarrow$ no equivalence between matrices $M(b)$ and $H(a)$.

Change of variable

Notation: primes: $p_1, p_2, \dots$; $n \in \mathbb{N}$
$n = p_1^{\varkappa_1} p_2^{\varkappa_2} \cdots =: p^{\varkappa}$, where $\varkappa = (\varkappa_1, \varkappa_2, \dots) \in \mathbb{N}_{\text{fin}}^{\infty}$.
Then for $n = p^{\varkappa}$ and $n' = p^{\varkappa'}$,

$$M(b) = \{b(nn')\}_{n,n' \in \mathbb{N}} = \{b(p^{\varkappa+\varkappa'})\}_{\varkappa,\varkappa' \in \mathbb{N}_{\text{fin}}^{\infty}}.$$

Example: $b(p^{\varkappa}) = b_1(\varkappa_1) \cdots b_d(\varkappa_d)$; then
$b(p^{\varkappa+\varkappa'}) = b_1(\varkappa_1 + \varkappa_1') \cdots b_d(\varkappa_d + \varkappa_d')$, so

$$M(b) = H(b_1) \otimes \cdots \otimes H(b_d).$$

- Helson matrices are more complex
- Number theoretic aspects

Hankel and Helson: function spaces

Hankel matrices $H(a) = \{a(n+m)\}_{n,m\geq 0}$:

Hardy space $H^2(\mathbb{T}) = \{f(z) = \sum_{n\geq 0} f_n z^n : \sum_n |f_n|^2 < \infty\}$, $|z| = 1$

$$\sum_{n,m\geq 0} a(n+m) f_n \overline{g_m} = \Big\langle \underbrace{\sum_{k\geq 0} a(k) z^k}_{\text{symbol}}, \Big(\sum_{n\geq 0} \overline{f_n} z^n\Big)\Big(\sum_{m\geq 0} g_m z^m\Big) \Big\rangle_{H^2(\mathbb{T})}$$

Helson matrices $M(b) = \{b(nm)\}_{n,m\geq 1}$:

Hardy space $\mathcal{H}^2$ of Dirichlet series

$$\mathcal{H}^2 = \Big\{F(s) = \sum_{n\geq 1} F_n n^{-s} : \sum_{n\geq 1} |F_n|^2 < \infty\Big\}, \quad \operatorname{Re} s > 1/2$$

$$\sum_{n,m\geq 1} b(nm) F_n \overline{G_m} = \Big\langle \underbrace{\sum_{k\geq 1} b(k) k^{-s}}_{\text{Dirichlet symbol}}, \Big(\sum_{n\geq 1} \overline{F_n} n^{-s}\Big)\Big(\sum_{m\geq 1} G_m m^{-s}\Big) \Big\rangle_{\mathcal{H}^2}$$

Hardy spaces of Dirichlet series

$F(s) = \sum_{n\geq 1} F_n n^{-s}$

$$\|F\|_{\mathcal{H}^2}^2 = \sum_{n\geq 1} |F_n|^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F(it)|^2 dt$$

$$\mathcal{H}^p : \quad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F(it)|^p dt$$

$$\mathcal{H}^\infty : \quad \sup_{\operatorname{Re} s>0} |F(s)| < \infty.$$

$\mathcal{H}^2$ introduced and studied by Hedenmalm-Lindqvist-Seip 1997
$\mathcal{H}^p$, $p \neq 2$ introduced and studied by Bayart 2002
A.Aleman, O.Brevig, A.Harper, H.Helson, S.Konyagin, J.-F.Olsen, J.Ortega-Cerda, K.Perfekt, H.Queffelec, E.Saksman

Bohr's correspondence

H. Bohr 1913: $$F(s) = \sum_{n\geq 1} f_n n^{-s} \quad \sim \quad f(z) = \sum_{\varkappa \in \mathbb{N}^\infty_{\mathrm{fin}}} f_{p^\varkappa} z^\varkappa,$$

$z = (z_1, z_2, \dots) \in \mathbb{D}^\infty$ – infinite polydisk, $z^\varkappa = z_1^{\varkappa_1} z_2^{\varkappa_2} \dots$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F(it)|^p dt = \int_{\mathbb{T}^\infty} |f(z)|^p dm_\infty(z)$$

$$\mathcal{H}^p = H^p(\mathbb{T}^\infty), \quad 1 \leq p \leq \infty.$$

For a Helson matrix $M(b) = \{b(nm)\}_{n,m\in\mathbb{N}}$
Dirichlet symbol: $\sum_n b(n) n^{-s}$
Infinite polydisk symbol: $\sum_\varkappa b(p^\varkappa) z^\varkappa$

Boundedness: Nehari's theorem

$H(a)$, analytic symbol: $\mathbf{a}(z) = \sum_{n\geq 0} a(n)z^n$, $z \in \mathbb{T}$.

$$P_+ : L^2(\mathbb{T}) \to H^2(\mathbb{T}), \quad \sum_{n\in\mathbb{Z}} f_n z^n \mapsto \sum_{n\geq 0} f_n z^n.$$

Theorem (Nehari 1957). *$H(a)$ is bdd iff there is a bdd symbol: $\exists A \in L^\infty(\mathbb{T})$ with $\mathbf{a} = P_+(A)$ (i.e. $\mathbf{a} \in \mathrm{BMO}$).*

$M(b)$, analytic symbol: $\mathbf{b}(z) = \sum_{\varkappa\in\mathbb{N}_{\mathrm{fin}}^\infty} b(p^\varkappa)z^\varkappa$, $z \in \mathbb{T}^\infty$.

$$P_+ : L^2(\mathbb{T}^\infty) \to H^2(\mathbb{T}^\infty), \quad \sum_{\varkappa\in\mathbb{Z}^\infty} F_\varkappa z^\varkappa \mapsto \sum_{\varkappa\in\mathbb{N}_{\mathrm{fin}}^\infty} F_\varkappa z^\varkappa.$$

Theorem (Ortega-Cerdà-Seip 2012). *$\exists$ a bdd Helson matrix $M(b)$ with no bdd symbol: $\nexists B \in L^\infty(\mathbb{T}^\infty)$ with $\mathbf{b} = P_+(B)$.*

What is the class of b corresponding to bounded $M(b)$???

Bounded symbols and Schatten classes

Is there a natural subclass of $M(b)$ with bdd symbols???

Theorem (Helson 2006). *If $M(b)$ is Hilbert-Schmidt, $M(b) \in \mathbf{S}_2$, then there exists a bounded symbol B.*

- Computing Hilbert-Schmidt norm:

$$\|M(b)\|_{\mathbf{S}_2}^2 = \sum_{n,m\geq 1} |b(nm)|^2 = \sum_{n\geq 1} |b(k)|^2 d(k),$$

  $d(k)$ is the number of divisors of k.

- Helson's inequality:

$$\Big(\sum_{n\geq 1} |F_n|^2/d(n)\Big)^{1/2} \leq \|F\|_{\mathcal{H}^1}, \quad F(s) = \sum F_n n^{-s}.$$

Is condition $M(b) \in \mathbf{S}_2$ optimal???

Brevig-Perfekt 2015: $M(b) \in \mathbf{S}_p$, $p > 1/(1 - \log \pi / \log 4) \approx 5.74$, is not sufficient for existence of bounded symbol.

The Hilbert matrix

$$H(a) = \{(1+n+m)^{-1}\}_{n,m\in\mathbb{Z}_+} \quad \text{with } a(n) = (1+n)^{-1}.$$

$$\text{Symbol:} \quad \sum_{n\geq 0} \frac{z^n}{n+1} - \sum_{n\geq 1} \frac{\overline{z}^n}{n+1} \quad \text{bdd, jump at } z=1$$

$H(a)$ is bounded, not compact. Borderline:

- $a(n)n \to 0$, $n \to \infty \Rightarrow H(a)$ is compact;
- $a(n)n \to \infty$, $n \to \infty \Rightarrow H(a)$ is unbounded.

Theorem (M. Rosenblum 1958). *The spectrum of $H(a)$ is purely absolutely continuous with multiplicity one, $\sigma(H(a)) = [0, \pi]$.*

- Explicit diagonalisation of $H(a)$ is available in terms of Laguerre functions.
- Used by S. Power (1982) to analyse Hankel operators with piecewise continuous symbols

The Multiplicative Hilbert matrix

$M(b) = \{(nm)^{-1/2}(\log(nm))^{-1}\}_{n,m\geq 2}$, $b(n) = n^{-1/2}(\log n)^{-1}$.
$M(b)$ is bounded and non-compact. Borderline:

- $b(n)n^{1/2}\log n \to 0 \Rightarrow M(b)$ is compact;
- $b(n)n^{1/2}\log n \to \infty \Rightarrow M(b)$ is unbounded.

Dirichlet symbol:

$$\sum_{n=2}^{\infty} b(n)n^{-s} = \sum_{n=2}^{\infty} n^{-\frac{1}{2}-s}(\log n)^{-1} = \int_{s_0}^{s} \zeta(\tfrac{1}{2}+s')ds' + \mathrm{const}.$$

$\exists$ a bounded symbol???
Theorem. *The spectrum of $M(b)$ is purely absolutely continuous with multiplicity one, $\sigma(M(b)) = [0,\pi]$.*

- Brevig-Perfekt-Seip-Siskakis-Vukotić 2016; Perfekt-P. 2017
- Explicit diagonalisation???

Further work

- Characterisation of finite rank Helson matrices:
  Perfect-P. 2017
  Finite rank Hankel matrices $\leftrightarrow$ rational symbols;
  Analogue of rational functions on $\mathbb{D}^\infty$.
- Characterization of positive semi-definite Helson matrices through moment problem
- Eigenvalue asymptotics of some Helson matrices: Miheisi-P. 2017

Conclusion

Hankel $H(a) = \{a(n+m)\}$ versus Helson $M(b) = \{b(nm)\}$

- Hankel: classical; Helson: last $\approx$10 years
- Helson: strong links to analytic number theory
- Hankel: $H^2(\mathbb{T})$; Helson: $H^2(\mathbb{T}^\infty)$
- $\Rightarrow$ classical methods often insufficient for Helson
- Some classical facts fail for Helson (Nehari)
- Many remaining challenges

THANK YOU!