Furuta's inequality and p-$wA(s,t)$ operator

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The aim

The aim of this talk is to speak about a small history with Furata's inequality and related classes of operator, that is, p-hyponormal operator, class $A(s,t)$ operator, class $wA(s,t)$ and p-$wA(s,t)$ operator where $0 < p \leq 1$ and $0 < s, t$.

References

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Motivation

Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. If $TT^* = T^*T$, T is called normal. Then T admits a spectral decomposition $T = \int_{\sigma(T)} \lambda dE(\lambda)$, and we can caluculate $f(T) = \int_{\sigma(T)} f(\lambda)dE(\lambda)$ for a continuous function $f(\lambda)$ on $\sigma(T)$. Hence normal operator is standard. However there are many non-normal operators.

Motivation

1. $T \in B(\mathcal{H})$ is called hyponormal, if $TT^* \leq T^*T$. Many mathematicians studied hyponormal operator (Xia, Putnam, Stampfli, ..). Hyponormal operator does not admit spectral decomposition, but has several interesting properties.

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2. Our motivation is to find new generalization of hyponormal operator and study its spectral properties.

Furuta's inequality(1987)

1. Let $0 < p, q, r$ satisfy
   $p + 2r \leq (1 + 2r)q$ and $1 \leq q$. (Furuta's area)
   If $O \leq B \leq A$, then
   $B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}}$ and $(A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}}$.

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2. Furuta's area was mysterious to me. So I asked Furuta, how do you find this area?

1. If $r = 0$, then $p + 2r \leq (1 + 2r)q$ means

$$p \leq q \text{ or } 0 < \frac{p}{q} \leq 1$$

and $B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}}$ means

$$B^{\frac{p}{q}} \leq (B^0 A^p B^0)^{\frac{1}{q}} = A^{\frac{p}{q}}.$$

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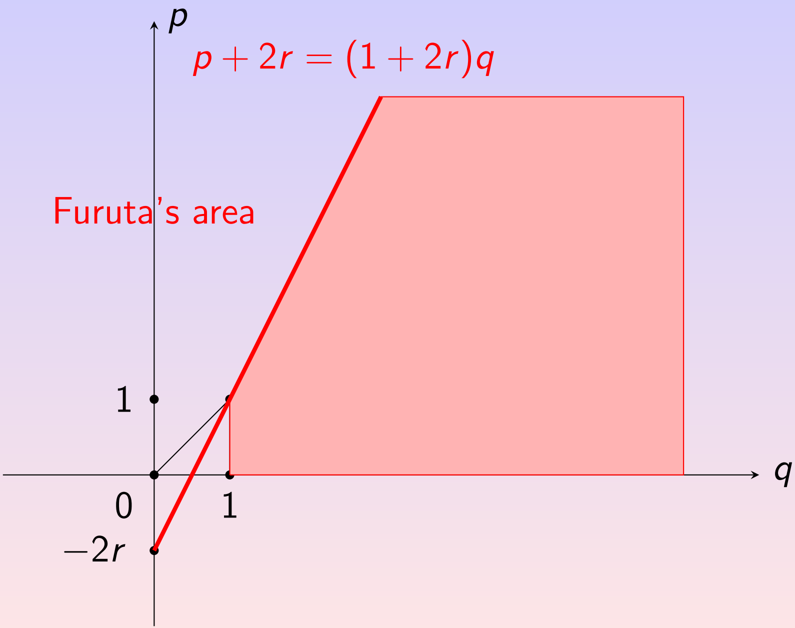
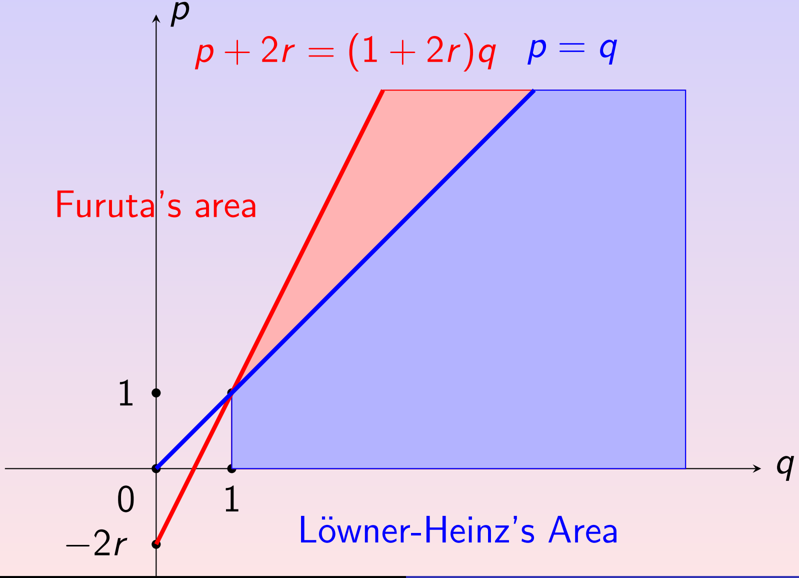
2. Hence Furuta's inequality is an extension of Löwner-Heinz's inequality.

$$0 \leq B \leq A \text{ and } 0 < p \leq 1 \Longrightarrow B^p \leq A^p.$$

Red domain is Furuta's area.



Blue domain is Löwner-Heinz's area. So Furuta extends Löwner-Heinz's inequality.



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3. If $p = 1$, then T is hyponormal $TT^* \leq T^*T$. Hence p-hyponormal operator is a generalization of hyponormal operator.

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Proposition. Let A_p be the set of all p-hyponormal operators. If $0 < q < p$, then $A_p \subset A_q$.

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Remark. Hence parameterized operator class A_p is increasing when $1 \geq p \to +0$.

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2. $\|T\| = r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$ normaloid.
3. $\pi \|T^*T - TT^*\| \leq \int_{\sigma(T)} r dr d\theta = \mathrm{meas}\ \sigma(T)$

   Hence if $\mathrm{meas}\ \sigma(T) = 0$, then T is normal. Putnam's inequality.

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2. $\|T\| = r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$ normaloid.
3. $\pi\|(T^*T)^p - (TT^*)^p\| \leq p \int_{\sigma(T)} r^{2p-1} dr d\theta$

   Hence if $\mathrm{meas}\, \sigma(T) = 0$, then T is normal. Putnam's inequality.

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3. $\left(T(\frac{1}{2}, \frac{1}{2})T(\frac{1}{2}, \frac{1}{2})^*\right)^{p+\frac{1}{2}} \leq \left(T(\frac{1}{2}, \frac{1}{2})^*T(\frac{1}{2}, \frac{1}{2})\right)^{p+\frac{1}{2}}$ Hence $T(\frac{1}{2}, \frac{1}{2})$ is $(p + \frac{1}{2})$-hyponormal and $T^2(\frac{1}{2}, \frac{1}{2})$ is hyponormal by Furuta's inequality

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2. If $T = U|T|$ is p-hyponormal (where small p as $0 < p < s, t$), then $T(s,t) = |T|^s U|T|^t$ satisfies

$$|T(s,t)|^{\frac{2(p+t)}{s+t}} \geq |T|^{2(p+t)}, |T|^{2(p+s)} \geq |T(s,t)^*|^{\frac{2(p+s)}{s+t}}$$

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3. We can take $p = 0$ by Löwner-Heinz's inequality. Then

class $A(s,t)$, $wA(s,t)$ oprator (1998)

1. $|T(s,t)|^{\frac{2t}{s+t}} \geq |T|^{2t}, |T|^{2s} \geq |T(s,t)^*|^{\frac{2s}{s+t}}$

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1. $|T(s,t)|^{\frac{2t}{s+t}} \geq |T|^{2t}, |T|^{2s} \geq |T(s,t)^*|^{\frac{2s}{s+t}}$
2. Ito, Furuta, Yamazaki(1998) defined that T is class $wA(s,t)$ operator if

$$|T(s,t)|^{\frac{2t}{s+t}} \geq |T|^{2t}, |T|^{2s} \geq |T(s,t)^*|^{\frac{2s}{s+t}}$$

and class $A(s,t)$ operator if

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2. So class $wA(s,t)$ is class $A(s,t)$, now.

class p-$wA(s,t)$ oprator (2014)

1. We define that T is class p-$wA(s,t)$ operator if

$$|T(s,t)|^{\frac{2pt}{s+t}} \geq |T|^{2pt}, |T|^{2ps} \geq |T(s,t)^*|^{\frac{2ps}{s+t}}$$

for $0 < p \leq 1$ and $0 < s, t \leq 1$.

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   for $0 < p \leq 1$ and $0 < s, t \leq 1$.
2. Hence p-$w(s,t)$ is a generalization of $wA(s,t)$. We assert (hope) p-$wA(s,t)$ is good generalization.

Properties of p-$wA(s, t)$ operator

1. If $0 < p_1 < p_2 \leq 1, 0 < s_2 < s_1, 0 < t_2 < t_1$, then class p_2-$wA(s_2, t_2)$ operator is class p_1-$wA(s_1, t_1)$.

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3. $\|T\| = r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$ normaloid

4. Putnam (type) inequality

$$\left\| |T(s,t)|^{\frac{2\min\{sp,tp\}}{s+t}} - |(T(s,t))^*|^{\frac{2\min\{sp,tp\}}{s+t}} \right\| \leq \frac{\min\{sp,tp\}}{\pi} \iint_{\sigma(T)} r^{2\min\{sp,tp\}-1}\, dr d\theta.$$

Moreover, if $\mathrm{meas}\,(\sigma(T)) = 0$, then T is normal.

Problem

Question. If T is class p-$wA(s,t)$ and $\mathcal{M}$ is T-invariant, then $T|_{\mathcal{M}}$ is p-$wA(s,t)$?

photo of Furuta (2004)

Thank you.