Fine scales of decay and an application to decay of waves in a viscoelastic boundary damping model

(International Workshop on Operator Theory and its Applications)

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(joint with Jan Rozendaal and David Seifert)

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Theory

General bounds for semiuniform decay

Mainly motivated by the wave equation in the past decade there has been much activity in semiuniform decay of C_0-semigroups (Batkai, Batty, Borichev, Chill, Duyckaerts, Engel, Liu, Martinez, Prüss, Rao, Rozendaal, Schnaubelt, D. Seifert, Stahn, Tomilov, Veraar). A famous result is the following:

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Theorem (Batty-Duyckaerts 2008)

Let $-A$ be the generator of a bounded C_0-SG T on a Banach space X with $\sigma(A) \cap i\mathbb{R} = \emptyset$. For $s \geq 0$ let

$$M(s) := \sup_{|\xi| \leq s} \left\| (i\xi + A)^{-1} \right\|.$$

Let $M_{log}(s) = M(s) \log(2 + s + M(s))$. Then

$$\forall t > 0 : \frac{c}{M^{-1}(c_2 t)} \leq \left\| T(t) A^{-1} \right\| \leq \frac{C}{M_{log}^{-1}(c_1 t)}$$

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Our aim: To find all admissible resolvent growth bounds M allowing to replace M_{log} by M in Hilbert spaces.

Admissible resolvent growth bounds

Definition

We call a non-decreasing function $M : [0, \infty) \to (0, \infty)$ admissible if for all bounded C_0-SGs $T \sim -A$ on Hilbert spaces with $\sigma(A) \cap i\mathbb{R} = \emptyset$ and

$$\forall s \geq 0 : \sup_{|\xi| \leq s} \left\| (i\xi + A)^{-1} \right\| \leq C_1 M(s)$$

it holds that

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Remark

We will see that M admissible implies $M^{-1}(ct) \approx M^{-1}(t)$ for all $c > 0$.

Our main result

Theorem (Rozendaal-Seifert-Stahn 2017)

*A non-decreasing function $M : [0,\infty) \to (0,\infty)$ is admissible if and only if it has positive increase ($M \in$ **PI**), that is:*

$$\exists \lambda > 1 : \liminf_{s\to\infty} \frac{M(\lambda s)}{M(s)} > 1$$

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The condition $M \in \mathbf{PI}$ is equivalent to

$$\exists \rho, s_0 > 0, b \in (0, 1] \forall s_0 \leq s \leq R : \frac{M(R)}{M(s)} \geq b \left(\frac{R}{s} \right)^{\rho}.$$

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Remark

Necessity of $M \in \mathbf{PI}$ for all normal semigroups.

Sufficiency of $M \in \mathbf{PI}$

(a) Fix $x \in D(A)$ and $t > 0$, let $g(\tau) = 1_{[0,t]}(\tau)T(\tau)x$ and write

$$g(t) = \frac{n+1}{t^{n+1}} \int_0^t s^n T(t-s)T(s)ds = \frac{(n+1)!}{t^{n+1}} \int_0^t T(t-s)T^{*n} * g(s)ds.$$

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(b) A truncation $(\delta - \phi_R) * T^{*n} * + \phi_R * T^{*n} *$ allows to treat the second term as a Fourier multiplier on $L^2(\mathbb{R}; L(D(A), X))$. The First term can be estimated by C/R.

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(c) Crucial in the estimation of the second term is the inequality (for large n)

$$R \sup_{|\xi| \le R} \left\| (i\xi - A)^{-n} \right\|_{L(D(A),X)} \le R \sup_{1 \le s \le R} s^{-1} M(s)^n \le b^{-n} M(R)^n.$$

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Only here we use $M \in \mathbf{PI}$.
(d) Optimization of the two estimates with respect to R finally yields the optimal decay rate.

Application

A model for sound reflection

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. The "velocity potential" U satisfies

$$\begin{cases} U_{tt}(t,x) - \Delta U(t,x) &= 0 \quad (t \in \mathbb{R}, x \in \Omega), \\ \partial_n U(t,x) + k * U_t(t,x) &= 0 \quad (t \in \mathbb{R}, x \in \partial\Omega). \end{cases}$$

Pressure $p = U_t$, fluid velocity $v = -\nabla U$. Here $k \in L^1(0,\infty)$ is completely monotonic, i.e. there exists a Radon measure $\nu \geq 0$ s.t. $k(t) = \int_0^\infty e^{-\tau t} d\nu(\tau)$.

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Theorem (Desch-Fasangova-Milota-Probst 2010, Stahn 2017)

(i) The operator $-\mathcal{A}$ generates a C_0-semigroup of contractions. Moreover A is injective and $\sigma(-\mathcal{A}) \cap i\mathbb{R} \subseteq \{0\}$.
(ii) $\exists s_0 > 0 \forall |s| \leq s_0 : \left\|(is + \mathcal{A})^{-1}\right\| \leq C|s|^{-1}$.
(iii) $\mathcal{A}$ is invertible iff $\exists \varepsilon > 0 : \nu([0,\varepsilon)) = 0$.

Bound on resolvent in terms of acoustic impedance

The 1D setting allows to explicitly calculate the resolvent of $\mathcal{A}$.

Theorem (Stahn 2017)

Let $\Omega = (0,1)$. Then for all $s \geq 1$

$$\frac{c}{\Re \hat{k}(is)} \leq \sup_{1 \leq |\xi| \leq s} \left\| (i\xi + \mathcal{A})^{-1} \right\| \leq \frac{C}{\Re \hat{k}(is)}.$$

Moreover the spectrum determines the resolvent growth.

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Under mild additional assumptions on Ω and $\hat{k}$ one can prove the upper bound also in higher dimensions. The proof is now based on recently proved trace properties of Laplace-Neumann eigenfunctions of Ω (see [Barnett-Hassel-Tacy 2016]).

Decay rates in terms of acoustic impedance

Corollary

Let $\Omega = (0,1)$ *assume* $\exists \varepsilon > 0 : \nu([0,\varepsilon)) = 0$ *and define* $M(s) = (\Re \hat{k}(is))^{-1}$*. Then*

$$\forall t \geq 1 : \left\| T(t)\mathcal{A}^{-1} \right\| \leq \frac{C}{M^{-1}(ct)}$$

holds for some $c, C > 0$ *if (and only if)* $1/\Re \hat{k}(i\cdot) \in$ **PI**.

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Proposition

Let $\alpha \in (0,2)$ and $l : \mathbb{R}_+ \to (0,\infty)$ be a slowly varying function. Then one can choose ν in such a way that $\nu|_{[0,1)} = 0$ and

$$\Re\hat{k}(is)^{-1} \sim s^{\alpha} l(s)$$

as $s \to \infty$.

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Viscoelastic boundary damping:
[4] Desch, Fasangova, Milota, Probst. *Stabilization through viscoelastic boundary damping: a semigroup approach*. Semigroup Forum 2010.
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See also: A. Benaissa et. al.; B. Mbodje; J. Prüss.

Thank you for your attention!