Best constants in Markov-type inequalities with mixed Hermite weights

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Introduction

We consider the inequalities of the form

$$\|D^\nu f\| \le C\|f\| \quad \text{for all } f \in \mathcal{P}_n,$$

where

- $\mathcal{P}_n$ ... algebraic polynomials of degree at most n,
- $D^\nu f$ ... νth derivative of f.

Question

What is the best (smallest) constant C such that the inequality holds for all $f \in \mathcal{P}_n$?

Introduction

The constant depends on n, ν, and the chosen norms. In this talk, we focus ourselves on the weighted Hermite norm

$$\|f\|_\alpha^2 = \int_{-\infty}^{\infty} |f(t)|^2 |t|^{2\alpha} \mathrm{e}^{-t^2} \mathrm{d}t,$$

and denote the best constant for

$$\|D^\nu f\|_\beta \le C \|f\|_\alpha \quad \text{for all } f \in \mathcal{P}_n$$

by $\eta_n^{(\nu)}(\alpha, \beta)$.

The story so far – the beginning

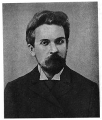
- D.I. Mendeleev (1880s):

$$\|D\|_\infty \le 4\|f\|_\infty \text{ for } f \in \mathcal{P}_2 \text{ on } [-1,1]$$

- A.A. Markov (1889):

$$\|Df\|_\infty \le n^2\|f\|_\infty \text{ for } f \in \mathcal{P}_n \text{ on } [-1,1]$$

- V.A. Markov (1892):



$$\|D^\nu f\|_\infty \le \frac{n^2(n^2-1)(n^2-2^2)\cdots(n^2-(\nu-1)^2)}{(2\nu-1)!!}\|f\|_\infty \text{ for } f \in \mathcal{P}_n \text{ on } [-1,1]$$

Image source: `www.math.technion.ac.il/hat`

The story so far – Hilbert space norms

- E. Schmidt (1944):

$$\lambda_n^{(1)}(0,0) \sim \frac{2}{\pi} n \qquad \text{(Laguerre)},$$
$$\gamma_n^{(1)}(0,0) \sim \frac{1}{\pi} n^2 \qquad \text{(Legendre/Gegenbauer)},$$
$$\eta_n^{(1)}(0,0) = \sqrt{2n} \qquad \text{(Hermite)}.$$

- P. Turán (1960):

$$\lambda_n^{(1)}(0,0) = \left(2 \sin \frac{\pi}{4n+2}\right)^{-1}$$

The story so far – higher derivatives

- L.F. Shampine (1965): $\lambda_n^{(2)}(0,0) \sim \frac{n^2}{\mu_0^2}$, $\gamma_n^{(2)}(0,0) \sim \frac{n^4}{4\mu_0^2}$,
$$\mu_0 \approx 1.8751041$$
(smallest positive solution of $1 + \cos\mu \cosh\mu = 0$)
- P. Dörfler (1987/90): bounds for $\lambda_n^{(\nu)}(0,0)$, $\nu \geq 3$
- A. Böttcher, P. Dörfler (2009): $\lambda_n^{(\nu)}(0,0) \sim \|K_\nu\|_\infty n^\nu$,
$$(K_\nu f)(x) = \frac{1}{(\nu-1)!} \int_0^x (x-y)^{\nu-1} f(y) \mathrm{d}y \quad \text{on } L^2(0,1)$$
- Böttcher, Dörfler (2010):
$$\lambda_n^{(\nu)}(\alpha,\alpha) \sim \|L_{\nu,\alpha}\|_\infty n^\nu,$$
$$\gamma_n^{(\nu)}(\alpha,\alpha) \sim \|G_{\nu,\alpha}\|_\infty n^{2\nu},$$

$L_{\nu,\alpha},\ G_{\nu,\alpha}$ ... integral operators on $L^2(0,1)$

The story so far – mixed weights

- Böttcher, Dörfler (2010/2011), motivated by a question of J. Prestin (let $\omega := \beta - \alpha - \nu$):

$$\lambda_n^{(\nu)}(\alpha,\beta) \sim \begin{cases} 2^\omega n^{(\beta-\alpha)/2} & : \omega \geq 0, \text{ integral}, \\ \left\| L^*_{\nu,\alpha,\beta} \right\|_\infty n^{(\nu-\omega)/2} & : \omega < -\frac{1}{2}, \end{cases}$$

$$\gamma_n^{(\nu)}(\alpha,\beta) \sim \begin{cases} n^\nu & : \omega \geq 0, \text{ integral}, \\ 2^\omega \left\| L^*_{\nu,\alpha,\beta} \right\|_\infty n^{\nu-\omega} & : \omega < -\frac{1}{2}, \text{ integral}, \end{cases}$$

with

$$(L^*_{\nu,\alpha,\beta} f)(x) = \frac{1}{\Gamma(-\omega)} \int_0^x x^{-\alpha/2} y^{\beta/2} (x-y)^{-\omega-1} f(y) \mathrm{d}y.$$

Determining the best constant

- Best constant is just the operator norm of the differential operator between the associated spaces.
- Investigate the matrix representation of this operator with respect to bases of orthonormal polynomials and determine its spectral norm:

$$\widehat{H}_k^{(\nu)}(t,\alpha) = \sum_{j=0}^{k-\nu} c_{jk}^{(\nu)}(\alpha,\beta)\widehat{H}_j(t,\beta),$$

$c_{jk}^{(\nu)}(\alpha,\beta)$: matrix entries,
$\widehat{H}_j(t,\beta)$: jth normalized Hermite polynomial for the parameter β.

Determining the best constant – matrix entries I

For $\quad w_k(\alpha)H_k^{(\nu)}(t,\alpha) = \sum_{j=0}^{k-\nu} c_{jk}^{(\nu)}(\alpha,\beta)w_j(\beta)H_j(t,\beta)$:

$$c_{jk}^{(\nu)}(\alpha,\beta) = 2^{k-j+\nu}\frac{w_k(\alpha)}{w_j(\beta)}\frac{\Gamma(\lfloor k/2\rfloor+1)}{\Gamma(\lfloor j/2\rfloor+1)}$$

$$\times\binom{\lceil (j+\nu)/2\rceil - 1/2}{\lfloor \nu/2\rfloor + \nu_k}\binom{\beta-\alpha-\lfloor \nu/2\rfloor-\nu_k}{(k-j-\nu)/2}$$

$$\times\,{}_3F_2\left[\begin{matrix}-\lfloor \nu/2\rfloor-\nu_k,\ -(k-j-\nu)/2,\ \beta+\lceil j/2\rceil+1/2\\ \beta-\alpha-\lfloor \nu/2\rfloor-\nu_k-(k-j-\nu)/2+1,\ \lceil j/2\rceil+1/2\end{matrix};1\right],$$

for $k-\nu-j\geq 0$ even, zero otherwise, where $\nu_k = k\nu \mod 2$ and

$$w_k(\alpha) = \left(2^k\sqrt{\Gamma(\lfloor k/2\rfloor+1)\Gamma(\lceil k/2\rceil+\alpha+1/2)}\right)^{-1}$$

(normalizing factor for the kth Hermite polynomial).

Determining the best constant – matrix entries II

The matrix has a chessboard structure. Assuming an even dimension, we can permutate the rows and columns in such a way that a block diagonal matrix is attained. We can work with those two blocks separately.
Slightly rewriting of those blocks yields, e.g.,

$$c^{(\nu)}_{2j,2k+\nu}(\alpha,\beta) =$$

$$2^{\nu}\Gamma(\lceil \nu/2 \rceil + 1)\sqrt{\frac{\Gamma(j+\beta+1/2)}{\Gamma(j+1)}\frac{\Gamma(k+\lfloor \nu/2 \rfloor+1)}{\Gamma(k+\lceil \nu/2 \rceil+\alpha+1/2)}} \quad \times$$

$$\sum_{\tau=0}^{\min\{\lceil \nu/2 \rceil, k-j\}} \binom{j+\lceil \nu/2 \rceil - 1/2}{\lceil \nu/2 \rceil - \tau}\binom{\beta+j+\tau-1/2}{\tau}\binom{\beta-\alpha-\lceil \nu/2 \rceil}{k-j-\tau}$$

Determining the best constant

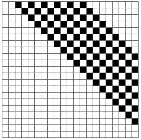
- In general an exact computation of the constant is not possible. Therefore, we investigate the asymptotic behaviour for $n \to \infty$.
- The methods for determining the spectral norm vary tremendously in dependence of the value of $\beta - \alpha$, due to the structure of the matrices.
- Essentially three possible cases:

$$\beta - \alpha \in \mathbb{N}_0, \quad \beta - \alpha \in (0, \infty) \setminus \mathbb{N}, \quad \beta - \alpha < -\tfrac{1}{2}$$

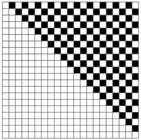
(and $\beta - \alpha \in \left[-\tfrac{1}{2}, 0\right)$).

Matrix structure

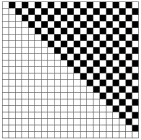
$\beta - \alpha \in \mathbb{N}_0$



$\beta - \alpha$
$\in (0, \infty) \setminus \mathbb{N}$



$\beta - \alpha < 0$



Matrix structure

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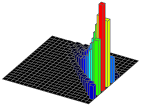
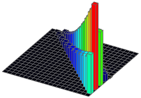
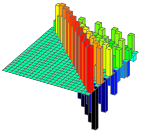


$\beta - \alpha$
$\in (0, \infty) \setminus \mathbb{N}$

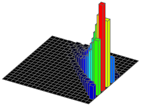


$\beta - \alpha < 0$



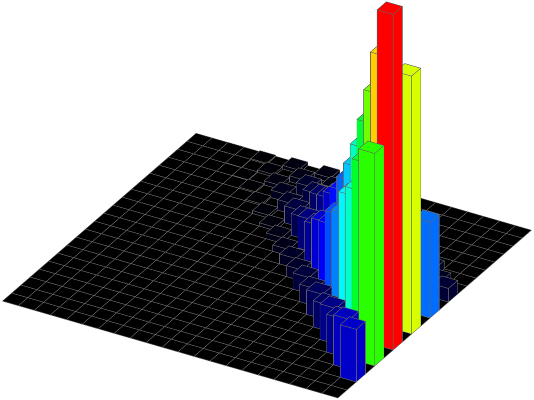
Matrix structure

$\beta - \alpha \in \mathbb{N}_0$



$\beta - \alpha \in (0, \infty) \setminus \mathbb{N}$

$\beta - \alpha < 0$



Matrix structure

$\beta - \alpha \in \mathbb{N}_0$

$\beta - \alpha$
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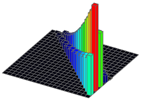
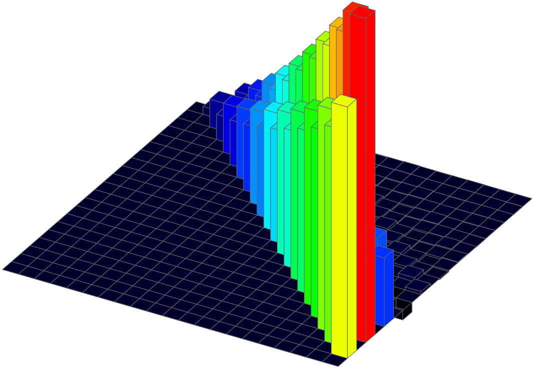


$\beta - \alpha < 0$

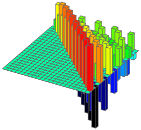


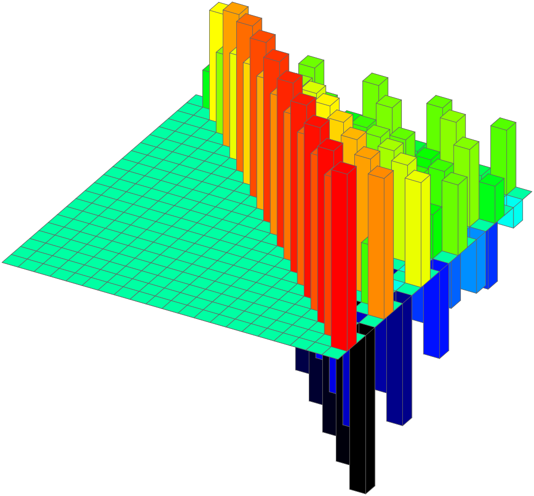
Matrix structure

$\beta - \alpha \in \mathbb{N}_0$

$\beta - \alpha \in (0, \infty) \setminus \mathbb{N}$

$\beta - \alpha < 0$





Integral case, $\beta - \alpha \geq \lceil \nu/2 \rceil$ or $\beta - \alpha = 0$

- Similar to Laguerre/Gegenbauer: banded matrix.
- Split matrix into diagonals to get upper estimate.
- Show convergence of (scaled) matrix to Toeplitz matrix to derive the lower bound.

Result

For integral $\beta - \alpha \geq \lceil \nu/2 \rceil$ or $\beta = \alpha$ we have

$$\eta_n^{(\nu)}(\alpha, \beta) \sim (2n)^{(\beta - \alpha + \nu)/2}$$

for $n \to \infty$.

Non-integral case, $\beta - \alpha > 0$, upper bound

To derive an upper bound, we interpolate the operator norm between the parameters shown. We use a lemma which is an application of a theorem of Stein. First, define the operator

$$T : L^2(\mathbb{R}, u(\cdot, \alpha)) \to L^2(\mathbb{R}, u(\cdot, \gamma)),$$

with $u(t, \alpha) = |t|^{2\alpha}\mathrm{e}^{-t^2}$, $\alpha, \gamma > -1/2$, like this:

$$\begin{aligned} Tf &= D^\nu f && \text{for all } f \in \mathcal{P}_n(\alpha), \\ Tg &= 0 && \text{for all } g \in \mathcal{P}_n(\alpha)^\perp, \end{aligned}$$

where $\mathcal{P}_n(\alpha)$ is the space of all algebraic polynomials, equipped with the norm

$$\|f\|_\alpha^2 = \int_{-\infty}^{\infty} |f(t)|^2 u(t, \alpha)\mathrm{d}t.$$

Non-integral case, $\beta - \alpha > 0$, upper bound

Lemma (L., 2016)

Under the aforementioned assumptions, we have for any $\gamma > -\frac{1}{2}$, $\|T\|_{\alpha\to\gamma} = \|D^\nu\|_{\alpha\to\gamma}$. If

$$\|D^\nu f\|_\beta \le C_n^{(\nu)} \|f\|_\alpha \qquad \text{for all } f \in \mathcal{P}_n \tag{$\star$}$$

is true for some $\beta = \beta'$ satisfying $\beta' - \alpha \in \mathbb{Z}$, and for all $\beta = \beta' + k$, $k \in K \subseteq \mathbb{N}$, K containing infinitely many numbers, and if $C_n^{(\nu)}(\alpha, \beta)$ satisfies

$$C_n^{(\nu)}(\alpha, \beta'(1-\theta)+(\beta'+k)\theta) = \left(C_n^{(\nu)}(\alpha, \beta')\right)^{1-\theta}\left(C_n^{(\nu)}(\alpha, \beta'+k)\right)^{\theta},$$

$\theta \in [0, 1]$, for all $k \in K$, then ($\star$) holds for all $\beta \in [\beta', \infty)$.

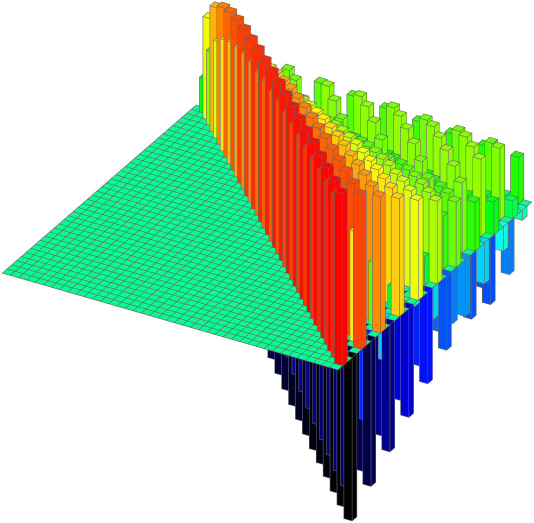
Non-integral case, $\beta - \alpha > 0$, lower bound

Deriving a lower bound is a bit more complex. The idea is to find a vector which

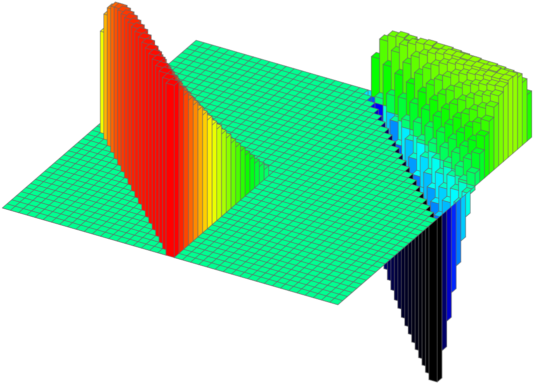
- provides a good bound on the norm,
- is easy to handle.

The main contribution to the norm comes from the entries alongside the main diagonal. After flipping the matrix, the largest parts are in the upper left block. Then, we choose the vector in such a way that only the first μ entries are non-zero and let μ grow with the dimension such that the fraction $(n - \mu)/n$ goes to one.
This way we get asymptotically the same result as for the upper bound.

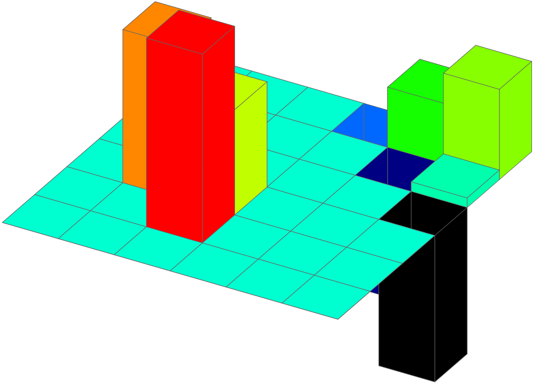
The case $\beta - \alpha < 0$



The case $\beta - \alpha < 0$

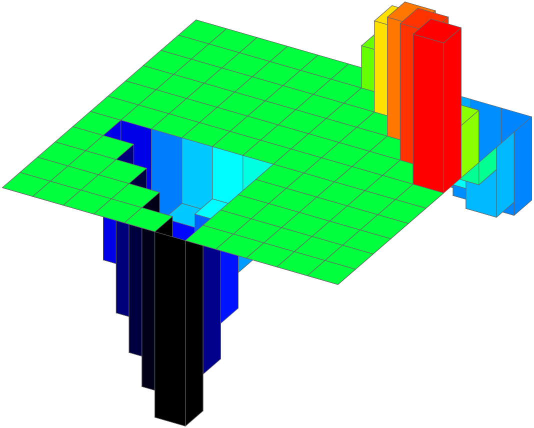


The case $\beta - \alpha < 0$



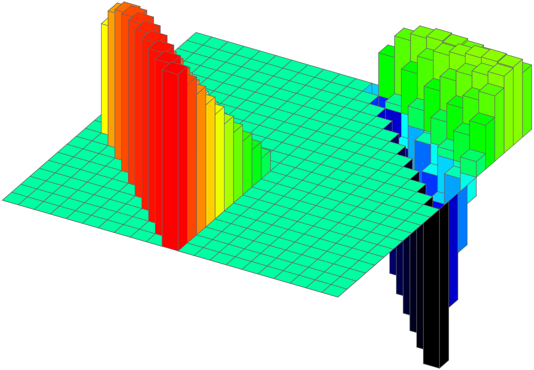
$n = 5$

The case $\beta - \alpha < 0$



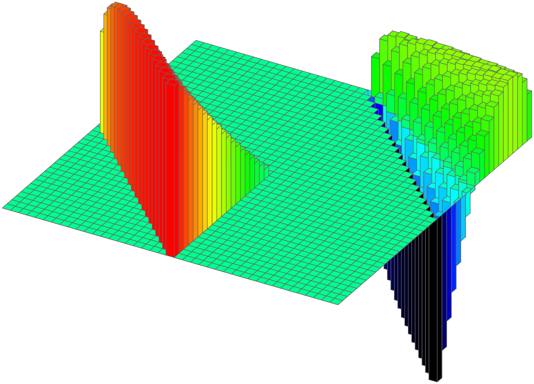
$n = 10$

The case $\beta - \alpha < 0$



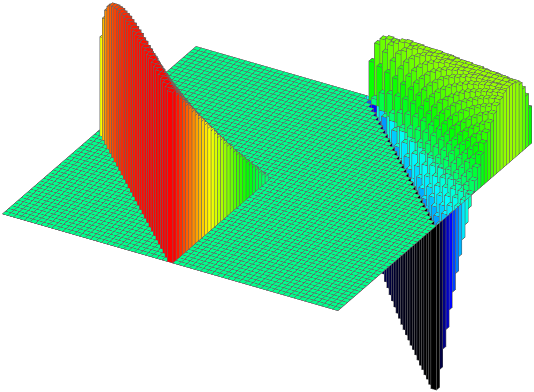
$n = 20$

The case $\beta - \alpha < 0$



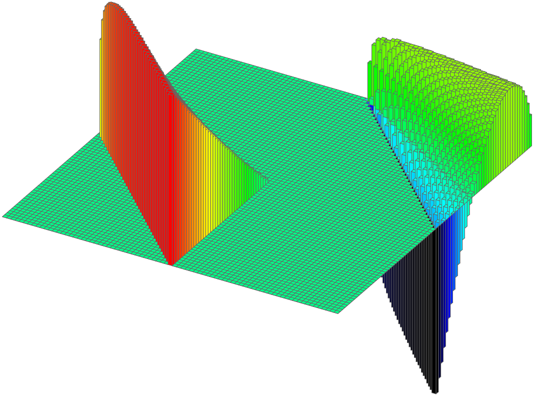
$n = 40$

The case $\beta - \alpha < 0$



$n = 60$

The case $\beta - \alpha < 0$



$n = 80$

The case $\beta - \alpha < 0$

Define an integral operator on $L^2(0,1)$ by

$$(K_N f)(x) = \int_0^1 k_N(x,y) f(y) \mathsf{d}y,$$

with $k_N(x,y) = (A_N)_{\lfloor Nx \rfloor, \lfloor Ny \rfloor}$, with A_N being the (modified) matrix.
Now we can use a result by Harold Widom which relates the norm of the matrix to the norm of the integral operator:

$$\|A_N\| = N\|K_N\|.$$

If we can show that the so defined operator converges in the norm to some integral operator, we get the desired result.

The case $\beta - \alpha < 0$

For $\beta - \alpha < -\frac{1}{2}$ the operator

$$\left(H^{(0)}_{\nu,\alpha,\beta} f\right)(x) = \frac{2^{\nu}\Gamma(\lceil \nu/2 \rceil + 1)}{\Gamma(\alpha - \beta + \lceil \nu/2 \rceil)} \times \int_x^1 x^{\beta/2-1/4} y^{-\alpha/2+1/4+(\lfloor \nu/2 \rfloor - \lceil \nu/2 \rceil)/2} (y-x)^{\alpha-\beta+\lceil \nu/2 \rceil+1} \times \sum_{\ell=0}^{\lceil \nu/2 \rceil} \binom{\beta}{\ell} \binom{\beta-\alpha-\ell}{\lceil \nu/2 \rceil - \ell} \left(\frac{x}{y-x}\right)^{\lceil \nu/2 \rceil - \ell} f(y)\mathrm{d}y$$

is Hilbert-Schmidt and we can show the convergence in the Hilbert-Schmidt norm and thus in the operator norm.

Conclusion

We have shown that for $\alpha, \beta > -1/2$, the following holds:

$$\eta_n^{(\nu)}(\alpha, \beta) \sim C_\nu(\alpha, \beta) n^{(|\beta-\alpha|+\nu)/2}$$

with

$$C_\nu(\alpha, \beta) = \begin{cases} 2^{(\beta-\alpha+\nu)/2} & : \beta - \alpha \geq 0, \\ 2^{(\beta-\alpha-\nu)/2} \cdot \max\left\{ \left\| H_{\nu,\alpha,\beta}^{(0)} \right\|, \left\| H_{\nu,\alpha,\beta}^{(1)} \right\| \right\} & : \beta - \alpha < -\frac{1}{2}. \end{cases}$$