Oblique projections and applications to weighted Procrustes type problems in Hilbert spaces

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Procrustes problems

Let $\mathcal{H}$ be a separable Hilbert space, $A,\ B \in L(\mathcal{H})$, we consider the following family of problems:
Determine the existence of

$$\min_X \|AX - B\|, \quad \text{for } X \in \mathcal{F},$$

where $\mathcal{F}$ is a given subset of $L(\mathcal{H})$.

Typically, X is required to be unitary, or a partial isometry or the range or null space of X have to satisfy a given inclusion, and the norm may be any unitarily invariant norm in $\mathcal{H}$.

These problems are known as *Procrustes problems*.

Löwdin orthogonalization

Problem: Given a basis $\{f_1, \cdots, f_n\}$ of $\mathbb{C}^n$, find the *closest* orthonormal basis $\{e_1, \cdots, e_n\}$.

For example, we can minimize

$$\sum_i^n \|f_i - e_i\|^2.$$

for $\{e_1, \cdots, e_n\}$ any o.n.b.

This problem was solved by P.-O Löwdin (1947), in connection to problems arising in Quantum Chemistry.

Löwdin orthogonalization

In terms of matrices, the problem becomes:
For a fixed invertible matrix F, minimize

$$\|F - U\|_2, \quad \text{subject to } U^*U = I$$

where $\|\cdot\|_2$ is the Frobenius norm.

If $F = U_F|F|$ is the polar decomposition of F, this problem has a global minimum at $U = U_F$, and

$$\||F| - I\|_2 = \|F - U_F\|_2 \leq \|F - U\|_2, \quad \text{for every unitary } U,$$

Löwdin (1970), J.G. Aiken, J.A. Erdos, J.A. Goldstein (1980).

$\mathcal{F} = \{f_j\}_{j\geq 1} \subset \mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist $a, b > 0$ such that

$$a\|f\|^2 \leq \sum_{j\geq 1} |\langle f, f_j \rangle|^2 \leq b\|f\|^2, \quad \text{for every } f \in \mathcal{H}.$$

If we can take $a = b = 1$, then $\mathcal{F}$ is a *Parseval* frame. In this case $\mathcal{F}$ satisfies the Parseval identity

$$\sum_{j\geq 1} |\langle f, f_j \rangle|^2 = \|f\|^2, \quad \text{for every } f \in \mathcal{H}.$$

Symmetric approximation of frames

The *synthesis operator* of the frame $\mathcal{F}$ is the operator $F : \ell^2(\mathbb{N}) \to \mathcal{H}$, defined as

$$F(\{\alpha_j\}_{j\geq 1}) = \sum_{j\geq 1} \alpha_j f_j,$$

and the *analysis operator* is its adjoint $F^* : \mathcal{H} \to \ell^2(\mathbb{N})$,

$$F^* f = \{\langle f, f_j \rangle\}_{j\geq 1}.$$

The *frame operator* of $\mathcal{F}$ is defined as

$$S_{\mathcal{F}} = FF^*.$$

Symmetric approximation of frames

Then

$$S_{\mathcal{F}} f = FF^* f = \sum_{j \geq 1} \langle f, f_j \rangle f_j, \text{ for every } f \in \mathcal{H};$$

and the inequalities in (6) can be expressed as

$$a \cdot I \leq S_{\mathcal{F}} \leq b \cdot I.$$

Therefore, $S_{\mathcal{F}} \in GL(\mathcal{H})^+$ and, $S_{\mathcal{F}} = I$ for Parseval frames.
From the equalities

$$f = S_{\mathcal{F}} S_{\mathcal{F}}^{-1} f = \sum_{j \geq 1} \langle S_{\mathcal{F}}^{-1} f, f_j \rangle f_j,$$

we get the *reconstruction formula*

$$f = \sum_{j \geq 1} \langle f, S_{\mathcal{F}}^{-1} f_j \rangle f_j, \quad \text{for every } f \in \mathcal{H}.$$

In particular, for Parseval frames, we get

$$f = \sum \langle f, f_j \rangle f_j, \quad \text{for every } f \in \mathcal{H}.$$

Symmetric approximation of frames

Problem: Given a frame $\mathcal{F}$, find the *closest* Parseval frame $\mathcal{V}$.

$\mathcal{F} = \{f_j\}_{j\geq 1} \subset \mathcal{H}$ is a *frame* for the (closed) subspace $\mathcal{K}$ of $\mathcal{H}$ if $\mathcal{F}$ is a frame for the Hilbert space $\mathcal{K}$.
The frames $\{f_i\}_{i\in\mathcal{N}}$ and $\{g_i\}_{i\in\mathcal{N}}$ of the closed subspaces $\mathcal{K}$ and $\mathcal{L} \subseteq \mathcal{H}$, are *weakly similar* if there exists $T \in GL(\mathcal{K}, \mathcal{L})$ such that $T(f_i) = g_i$, for every $i \in \mathbb{N}$.
Given $\{f_i\}_{i\in\mathcal{N}}$, a frame of $\mathcal{K} \subseteq \mathcal{H}$, a Parseval frame $\{\nu_i\}_{i=1}^{\infty}$ of $\mathcal{L} \subseteq \mathcal{H}$, is a *symmetric approximation* of $\{f_i\}_{i\in\mathcal{N}}$, if the frames $\{f_i\}_{i\in\mathcal{N}}$ and $\{\nu_i\}_{i\in\mathcal{N}}$ are weakly similar, the sum

$$\sum_{j\geq 1} \|\nu_j - f_j\|^2 < \infty$$

and

$$\sum_{j\geq 1} \|\nu_j - f_j\|^2 \leq \sum_{j\geq 1} \|\mu_j - f_j\|^2$$

for any other finite sum, corresponding to any Parseval frame $\{\mu_i\}_{i=1}^{\infty}$ of any subspace of $\mathcal{H}$ weakly similar to $\{f_i\}_{i\in\mathcal{N}}$.

Symmetric approximation of frames

If F, V and U are the synthesis operators of $\{f_i\}_{i\in\mathcal{N}}$, $\{\nu_i\}_{i\in\mathcal{N}}$ and $\{\mu_i\}_{i\in\mathcal{N}}$, then $\{\nu_i\}_{i\in\mathcal{N}}$ is a symmetric approximation of $\{f_i\}_{i\in\mathcal{N}}$ if

$$\|F - V\|_2 \leq \|F - U\|_2,$$

for all partial isometries U, with $N(U) = N(F)$, (this condition is equivalent to saying that the frames $\{f_i\}_{i\in\mathcal{N}}$ and $\{\mu_i\}_{i\in\mathcal{N}}$ are weakly similar).

Symmetric approximation of frames

If $F = U_F|F|$ is the canonical polar decomposition, a symmetric approximation exists and it is unique if and only if $(P - |F|)$ is a Hilbert-Schmidt operator, where $P = P_{R(F^*F)}$, (M. Frank, V. Paulsen and R. Tiballi, 2002).
In this case

$$\||F| - P\|_2 = \|F - U_F\|_2 \leq \|F - U\|_2,$$

for every partial isometry U, weakly similar to F.
The frame corresponding to the frame operator U_F is called the *canonical Parseval frame* associated to $\{f_i\}_{i \in \mathcal{N}}$.

If we drop the weakly similarity condition, the canonical Parseval frame can fail to be the closest Parseval frame. Results in this direction were given by J. Antezana and E. Chiumento (2016).

Consistent Sampling

Consider $\mathcal{S}$, (*the sampling space*), and $\mathcal{R}$, (*the reconstruction space*), two closed subspaces of $\mathcal{H}$.

Given a frame $\{v_n\}_{n\in\mathbb{N}}$ of $\mathcal{S}$, with synthesis operator $B : \ell^2(\mathbb{N}) \to \mathcal{H}$, the *samples* of a *signal* $f \in \mathcal{H}$ are given by

$$\{f_n\}_{n\in\mathbb{N}} = \{\langle f, v_n \rangle\}_{n\in\mathbb{N}} = B^* f.$$

On the other hand, given *samples* $\{f_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, the *reconstructed signal* $\hat{f}$ is given by

$$\hat{f} = \sum_{n\in\mathbb{N}} f_n w_n = A(\{f_n\}_{n\in\mathbb{N}}),$$

where $\{w_n\}_{n\in\mathbb{N}}$, is a frame of $\mathcal{R}$, with synthesis operator $A : \ell^2(\mathbb{N}) \to \mathcal{H}$.

Consistent Sampling

$$
\begin{array}{lllll}
\mathit{SIGNAL} & \longrightarrow & \mathit{SAMPLES} & \longrightarrow & \mathit{RECOVERED\ \ SIGNAL} \\
f & \longrightarrow & \{\langle f, v_n \rangle\}_{n\in\mathbb{N}} & \longrightarrow & \hat{f} = \sum_{n\in\mathbb{N}} \langle f, v_n \rangle\, w_n \\
f & \longrightarrow & B^* f & \longrightarrow & \hat{f} = AB^* f
\end{array}
$$

Consistent Sampling

Sometimes, by applying a *filter* $X \in L(\ell^2(\mathbb{N}))$, we can obtain a *better* reconstruction $\hat{f} = AXB^* f$:
Classical sampling scheme ($\mathcal{S} = \mathcal{R}$): It is possible to find X such that $AXB^* = P_{\mathcal{S}}$, where $P_{\mathcal{S}}$ is the orthogonal projection onto $\mathcal{S}$. Then

$$\hat{f} = P_{\mathcal{S}} f.$$

Consistent sampling scheme ($\mathcal{S}$ and $\mathcal{R}$ may not coincide): We ask for

$$B^* \hat{f} = B^* f.$$

(The samples of the reconstructed signal and the samples of the original signal are equal). In this case $Q = AXB^*$ turns out to be an oblique projection, (Y.C. Eldar, T. Werther, 2005).
But

$$\|f - \hat{f}\| = \|f - AXB^* f\|$$

is not necessarily minimized.

Consistent Sampling

Problem: Find a *good approximation* of f in $\mathcal{R}$.
For instance, find a *filter* $X_0 \in L(\ell^2(\mathbb{N}))$ such that

$$\|(AX_0B^* - I)f\| \leq \|(AXB^* - I)f\|,$$

for every $X \in L(\mathcal{H})$ and every $f \in \mathcal{H}$.
Or equivalently, study the existence of

$$\min_{X \in L(\ell^2(\mathbb{N}))} (AXB^* - I)^*(AXB^* - I),$$

with the usual order in $L(\mathcal{H})$.
Alternatively, we can approximate in some convenient operator norm.
In the finite dimensional setting, it is usual to consider the Frobenius norm $\|\cdot\|_2$; the associated problem becomes studying the existence of

$$\min_{X \in L(\ell^2(\mathbb{N}))} \|AXB^* - I\|_2.$$

Background

G.R. Goldstein and J.A. Goldstein (2000) analyzed the existence of

$$\min_{X \in L(\mathcal{H})} \|AX - I\|,$$

for unitarily invariant norms in finite dimensional spaces; H.W. Engle and M.Z. Nashed, (1981), studied a similar problem for the Schatten norms, in Hilbert spaces.
G. Corach, P. Massey and M. Ruiz, (2014), studied the existence of

$$\min_{X \in L(\mathcal{H})} \|AX^* - I\|, \text{ subject to } XX^* = 1,$$

for the operator norm, in the context of frames and Parseval duals.
There are also some inconclusive results on the existence of

$$\min_{X \in L(\mathcal{H})} \|AXB - C\|_p,$$

in Hilbert spaces, under certain conditions.

Procrustes type problem

Sometimes, it is necessary to stress some of the sampling coordinates differently. To this end, a positive weight W, i.e. a positive operator, is introduced that gives rise to a semi-norm:
Let $W \in L(\mathcal{H})$ be a positive operator such that $W^{1/2} \in S_p$, the p-Schatten class, for some p with $1 \leq p < \infty$.
Given A, $B \in L(\mathcal{H})$, A with closed range, analyze the existence of

$$\min_{X \in L(\mathcal{H})} \|AXB - I\|_{p,W}, \tag{0.1}$$

where $\| \cdot \|_{p,W} = \| W^{1/2} \cdot \|_p$.

Procrustes type problem

Taking $\mathcal{S} = N(B)$, problem (0.1) can be restated as a Procrustes problem type:
Given $A \in L(\mathcal{H})$ with closed range and $\mathcal{S}$ a closed subspace of $\mathcal{H}$, analyze the existence of

$$\min_{X \in L(\mathcal{H})} \|AX - I\|_{p,W}, \qquad \text{subject to } \mathcal{S} \subseteq N(X).$$

Oblique projections

When a positive weight W is introduced in $\mathcal{H}$, it can be useful to consider *W-orthogonal* projections, with a suitable prescribed range $\mathcal{S}$:

A positive operator $W \in L(\mathcal{H})$ and a closed subspace $\mathcal{S}$ are compatible if there exists an oblique projection $Q \in L(\mathcal{H})$ onto $\mathcal{S}$, such that

$$WQ = Q^* W,$$

or equivalently, Q is W-selfadjoint (i.e. selfadjoint with respect to the semi-inner product associated to W: $\langle x, y \rangle_W = \langle Wx, y \rangle$).

Oblique projections

A projection Q onto $\mathcal{S}$ is W-selfadjoint if and only if $N(Q) \subseteq W(\mathcal{S})^{\perp}$. Therefore:
W and $\mathcal{S}$ are compatible if and only if

$$\mathcal{H} = \mathcal{S} + (WS)^{\perp}.$$

This sum is not necessarily direct, so there might be infinite W-selfadjoint projections onto $\mathcal{S}$.

Oblique projections

Let $W \in L(\mathcal{H})^+$, $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. Then TFAE:

i) W and $\mathcal{S}$ are compatible.

ii) $\sup\{|\langle x, y\rangle| : x \in \mathcal{S}^\perp, y \in \overline{W(\mathcal{S})}, \|x\| = \|y\| = 1\} < 1$, (an angle condition).

iii) The equation

$$P_{\mathcal{S}} W = P_{\mathcal{S}} W P_{\mathcal{S}} X,$$

admits a solution, where $P_{\mathcal{S}}$ is the orthogonal projection onto $\mathcal{S}$, (a range inclusion condition).

iv) $R(W + P_{\mathcal{S}^\perp}) = R(W) + \mathcal{S}^\perp$, (a range additivity condition).

G. Corach, A. M., D. Stojanof, (2001); L. Arias, G. Corach, A. M.,(2015).

Shorted operator

Let $W \in L(\mathcal{H})^+$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. The shorted operator $W_{/\mathcal{S}}$ is the *biggest* positive operator acting on $\mathcal{S}^\perp$, that can be subtracted to W, such that the difference remains positive. More precisely:
The shorted operator $W_{/\mathcal{S}}$ is given by

$$W_{/\mathcal{S}} = \max \{X \in L(\mathcal{H}) : \ 0 \leq X \leq W \text{ and } R(X) \subseteq \mathcal{S}^\perp\},$$

(M.G. Kreĭn, 1947).

Shorted operator

Let $W \in L(\mathcal{H})^+$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. Then

i)
$$W_{/\mathcal{S}} = \inf \{E^* W E : E^2 = E,\ N(E) = \mathcal{S}\};$$
in general, this infimum is not attained, (W.N. Anderson and G.E. Trapp, 1975).

ii) $R(W) \cap \mathcal{S}^\perp \subseteq R(W_{/\mathcal{S}}) \subseteq R(W^{1/2}) \cap \mathcal{S}^\perp$ and $N(W_{/\mathcal{S}}) = W^{-1/2}(\overline{W^{1/2}(\mathcal{S})})$.

Shorted operator and compatibility

Theorem

Let $W \in L(\mathcal{H})^+$ and $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace. TFAE:

i) W *and* $\mathcal{S}$ *are compatible,*

ii) $W_{/\mathcal{S}} = min\,\{E^*WE : E^2 = E,\ N(E) = \mathcal{S}\}$,

iii) $R(W_{/\mathcal{S}}) = R(W) \cap \mathcal{S}^\perp$ *and* $N(W_{/\mathcal{S}}) = N(W) + \mathcal{S}$.

In this case,

$$W_{/\mathcal{S}} = W(I - Q),$$

for any W-selfadjoint projection Q onto $\mathcal{S}$.

Corach, M., Stojanof, (2002).

Operator order minimization problem

To study Problem (0.1) we return to the associated problem:
Given A, B and $W \in L(\mathcal{H})$, where A is a closed range operator and W a positive operator, analyze the existence of

$$\min_{X \in L(\mathcal{H})} (AXB - I)^* W (AXB - I),$$

with the usual order in $L(\mathcal{H})$.

Operator order minimization problem

The following results are in a joint paper with M. Contino and J. Giribet.
Under certain hypothesis, the infimum of the set considered above always exists:

Proposition

Let $A, B \in CR(\mathcal{H})$ *and* $W \in L(\mathcal{H})^+$. *If* $N(B) \subseteq N(A^*W)$ *then the infimum of the set* $\{(AXB - I)^* W(AXB - I) : X \in L(\mathcal{H})\}$ *exists and*

$$\inf_{X \in L(\mathcal{H})} (AXB - I)^* W(AXB - I) = W_{/R(A)}. \qquad (0.2)$$

Operator order minimization problem

Existence of minimum:

Theorem

Let $A, B \in CR(\mathcal{H})$ and $W \in L(\mathcal{H})^+$. Then TFAE:

i) $\min\limits_{X \in L(\mathcal{H})} (AXB - I)^* W (AXB - I)$ *exists.*

ii) W *and* $R(A)$ *are compatible and* $N(B) \subseteq N(A^* W)$.

iii) *The normal equation* $A^* W (AXB - I) = 0$ *admits a solution.*

Operator order minimization problem

If any of these conditions holds, then

$$\min_{X \in L(\mathcal{H})} (AXB - I)^* W (AXB - I) = W_{/R(A)}$$

and the minimum is attained at the solutions of the *normal* equation

$$A^* W (AXB - I) = 0.$$

Procrustes type problem

Back to the original problem:
Given $A, B \in CR(\mathcal{H})$ and $W \in L(\mathcal{H})^+$ such that $W^{1/2} \in S_p$ for some p with $1 \le p < \infty$, analyze the existence of

$$\min_{X \in L(\mathcal{H})} \|AXB - I\|_{p,W}. \tag{0.3}$$

Recalling (0.2),

$$\inf_{X \in L(\mathcal{H})} (AXB - I)^* W (AXB - I) = W_{/R(A)},$$

and the fact that $A^*A \le B^*B$ implies $\|A\|_p \le \|B\|_p$ for operators in S_p, we have:

$$\inf_{X \in L(\mathcal{H})} \|AXB - I\|_{p,W} \ge \|W_{/R(A)}^{1/2}\|_p.$$

Procrustes type problem

Proposition

Let $A, B \in CR(\mathcal{H})$ and $W \in L(\mathcal{H})^+$, such that $W^{1/2} \in S_p$, for some p with $1 \leq p < \infty$.
If W and $R(A)$ are compatible and $N(B) \subseteq N(A^*W)$ then the minimum of problem (0.3) exists and

$$\min_{X \in L(\mathcal{H})} \|AXB - I\|_{p,W} = \|W^{1/2}_{/R(A)}\|_{p,W}.$$

Procrustes type problem

Lemma

Let $A, B \in CR(\mathcal{H})$ *and* $W \in L(\mathcal{H})^+$, *such that* $W^{1/2} \in S_p$ *for some* p *with* $1 < p < \infty$ *and consider* $F_p(X) = \|AXB - I\|_{p,W}^p$. *Then,* $X_0 \in L(\mathcal{H})$ *is a global minimum of* F_p *if and only if* $X_0 \in L(\mathcal{H})$ *is a solution of*

$$B|W^{1/2}(AXB - I)|^{p-1}U^* W^{1/2}A = 0,$$

where $W^{1/2}(AXB - I) = U|W^{1/2}(AXB - I)|$ *is the polar decomposition of the operator* $W^{1/2}(AXB - I)$, *with* U *a partial isometry with* $N(U) = N(W^{1/2}(AXB - I))$.

Procrustes type problem: general case

Theorem

Let $A, B \in CR(\mathcal{H})$, $C \in L(\mathcal{H})$ and $W \in L(\mathcal{H})^+$, such that $W^{1/2} \in S_p$ for some p with $1 \leq p < \infty$ and $N(B) \subseteq N(A^*WC)$. Then TFAE:

i) $\min_{X \in L(\mathcal{H})} \|AXB - C\|_{p,W}$ *exists.*

ii) *The normal equation* $A^*W(AXB - C) = 0$ *admits a solution.*

iii) $R(C) \subseteq R(A) + R(A)^{\perp_W}$.

iv) $\min_{X \in L(\mathcal{H})} (AXB - C)^*W(AXB - C)$ *exists.*

Procrustes type problem: general case

In this case,

$$\min_{X \in L(\mathcal{H})} \|AXB - C\|_{p,W} = \|W_{/R(A)}^{1/2} C\|_p.$$

Moreover,

$$\|AX_0B - C\|_{p,W} = \|W_{/R(A)}^{1/2} C\|_p,$$

if and only if

$$A^* W(AX_0B - C) = 0.$$

Procrustes type problem: $p = 2$

When $p = 2$, it is possible to characterize the existence of minimum of Problem (0.3), without additional assumptions.

Theorem

Let $A, B \in CR(\mathcal{H})$, $C \in L(\mathcal{H})$ and $W \in L(\mathcal{H})^+$, such that $W^{1/2} \in S_2$. Then TFAE:

i) $\min_{X \in L(\mathcal{H})} \|AXB - C\|_{2,W}$ *exists.*

ii) *The equation* $A^*W(AXB - C)B^* = 0$ *admits a solution.*

iii) $R(CB^*) \subseteq R(A) + R(A)^{\perp_W}$.

Final remarks

The existence of solutions of

$$\min_{X \in L(\mathcal{H})} (AXB - C)^* W (AXB - C),$$

implies the existence of solutions of

$$\min_{X \in L(\mathcal{H})} \|AXB - C\|_{p,W}.$$

But the converse is not true: in fact it easy to provide an example for matrices. Notice that if $N(B)$ is not included in $N(A^*WC)$ the first problem does not have a solution.

It can also be shown that, for $1 < p < \infty$, $p \neq 2$, a minimum of the second problem need not satisfy the normal equation

$$A^* W (AXB - C) B^* = 0.$$

¡Muchas Gracias!