Asymmetric truncated Toeplitz operators of rank one

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Classical Toeplitz operators

H^2	-	**the Hardy space** for the unit disc $\mathbb{D}$,
P	-	the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2,
T_φ	-	the classical Toeplitz operator on H^2:

$$T_\varphi f = P(\varphi f), \quad f \in H^\infty \subset H^2,$$

- densely defined for $\varphi \in L^2(\partial\mathbb{D})$,
- bounded if and only if $\varphi \in L^\infty(\partial\mathbb{D})$.

In particular,

$S = T_z$	-	**the shift operator** on H^2,
$S^* = T_{\bar{z}}$	-	**the backward shift**,

$$Sf(z) = zf(z), \qquad S^*f(z) = \frac{f(z) - f(0)}{z}.$$

An inner function/ADC

We say that α is an **inner function** if:

- $\alpha \in H^\infty$,
- $|\alpha| = 1$ a.e. on $\partial\mathbb{D}$.

We say that α has an **angular derivative in the sense of Carathéodory (ADC) at** $w \in \partial\mathbb{D}$ if there exist complex numbers $\alpha(w)$ and $\alpha'(w)$ such that

$$\alpha(z) \to \alpha(w) \in \partial\mathbb{D} \qquad \text{and} \qquad \alpha'(z) \to \alpha'(w)$$

whenever $z \to w$ nontangentially (with $|z - w|/(1 - |z|)$ bounded).

Shift invariant subspaces of H^2

All shift-invariant subspaces of H^2 were described by A. Beurling in 1949. He used the notion of an inner function.

Beurling, 1949

A non-zero closed subspace $M \subset H^2$ is S-invariant, $S(M) \subset M$, if and only if $M = \alpha H^2$ for some inner function α.

Since $S(M) \subset M$ if and only if $M = \alpha H^2$, then $S^*(M) \subset M$ if and only if $M = (\alpha H^2)^\perp$.

Corollary

All the S^*-invariant subspaces of H^2 are of the form

$$K_\alpha = (\alpha H^2)^\perp = H^2 \ominus \alpha H^2, \quad \alpha - \text{inner}.$$

K_α is called **the model space** corresponding to the inner function α.

The model space K_α

The model space corresponding to the inner function α:

$$K_\alpha = H^2 \ominus \alpha H^2.$$

- K_α is a closed S^*-invariant subspace of H^2.
- K_α is a reproducing kernel Hilbert space with the **reproducing kernel** given by:

$$k_w^\alpha(z) = \frac{1 - \overline{\alpha(w)}\alpha(z)}{1 - \overline{w}z}, \quad w \in \mathbb{D},$$

  that is,

$$f(w) = \langle f, k_w^\alpha \rangle \quad \text{for all } f \in K_\alpha,\ w \in \mathbb{D}.$$

  Note that if $\alpha(w) = 0$, then $k_w^\alpha(z) = k_w(z) = (1 - \overline{w}z)^{-1}$.
- $K_\alpha \cap H^\infty$ is a dense subset of K_α.

The model space K_α

- The **conjugate kernel**

$$\widetilde{k}_w^\alpha(z) = \frac{\alpha(z) - \alpha(w)}{z - w}$$

  belongs to K_α for all $w \in \mathbb{D}$.
- If α has an ADC at $w \in \partial\mathbb{D}$, then k_w^α and $\widetilde{k}_w^\alpha$ belong to K_α. Moreover, $\widetilde{k}_w^\alpha = \alpha(w)\overline{w}k_w^\alpha$.

Examples:

1. $\alpha(z) = z^n$, $n \geq 1$:

$$K_\alpha = \mathcal{P}_{n-1} = \{\text{polynomials of degree} \leq n-1\},$$

2. $\alpha(z) =$ a finite Blaschke product with distinct zeros $a_1, \ldots, a_n$:

$$K_\alpha = \text{span}\{k_{a_1}, \ldots, k_{a_n}\}.$$

The space K_α is finite-dimensional, $\dim K_\alpha = n < \infty$, if and only if α is a finite Blaschke product with n zeros (not necessarily distinct).

Let $\varphi \in L^2(\partial\mathbb{D})$.

The classical Toeplitz operator T_φ:

$$T_\varphi f = P(\varphi f), \quad f \in H^\infty \subset H^2,$$

P - the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2.

Let α and β be two inner functions. The **asymmetric truncated Toeplitz operator** (ATTO) $A_\varphi^{\alpha,\beta}$ is defined by

$$A_\varphi^{\alpha,\beta} f = P_\beta(\varphi f), \quad f \in K_\alpha \cap H^\infty,$$

P_β - the orthogonal projection from $L^2(\partial\mathbb{D})$ onto K_β.

In particular, $A_\varphi^\alpha = A_\varphi^{\alpha,\alpha}$ is called a truncated Toeplitz operator (TTO). The operator $S_\alpha = A_z^\alpha$ is called the **compressed shift**.

Systematic study of truncated Toeplitz operators was started by D. Sarason in 2007.

Asymmetric truncated Toeplitz operators were recently introduced by C. Câmara, J. Partington (for the half-plane) and C. Câmara, J. Jurasik, K. Kliś-Garlicka, M. Ptak (for the unit disk).

Put

$$\mathscr{T}(\alpha, \beta) = \{A_{\varphi}^{\alpha,\beta} \colon \varphi \in L^2(\partial\mathbb{D}) \text{ and } A_{\varphi}^{\alpha,\beta} \text{ is bounded}\}.$$

More on $\mathscr{T}(\alpha, \beta)$

Although similar in definition, TTO's and ATTO's differ from the classical Toeplitz operators.

- $T_\varphi = 0$ if and only if $\varphi = 0$,

Câmara-Partington/Câmara-Jurasik-Kliś–Garlicka-Ptak ($\beta \leq \alpha$), Ł.-Jurasik, 2016

$A_\varphi^{\alpha,\beta} = 0$ if and only if $\varphi \in \overline{\alpha H^2} + \beta H^2$.

- T_φ is bounded if and only if φ is in $L^\infty(\partial\mathbb{D})$,

Baranov-Chalendar-Fricain-Mashreghi-Timotin, 2010

There exist bounded truncated Toeplitz operators without bounded symbols.

- the only compact Toeplitz operator is the zero operator,

Sarason, 2007

There are nonzero compact truncated Toeplitz operators (in particular, rank-one truncated Toeplitz operators).

Rank-one TTO's

Rank-one TTO's were described by D. Sarason.
Recall that

$$k_w^\alpha(z) = \frac{1-\overline{\alpha(w)}\alpha(z)}{1-\overline{w}z}, \quad \widetilde{k}_w^\alpha(z) = \frac{\alpha(z)-\alpha(w)}{z-w}$$

and $f \otimes g(h) = \langle h, g \rangle f$.

Sarason, 2007

(a) For w in $\mathbb{D}$, the operators $k_w^\alpha \otimes \widetilde{k}_w^\alpha$ and $\widetilde{k}_w^\alpha \otimes k_w^\alpha$ belong to $\mathscr{T}(\alpha,\alpha)$.

(b) If α has an ADC at the point w of $\partial\mathbb{D}$, then the operator $k_w^\alpha \otimes k_w^\alpha$ belongs to $\mathscr{T}(\alpha,\alpha)$.

(c) The only rank-one operators in $\mathscr{T}(\alpha,\alpha)$ are the nonzero scalar multiples of the operators in (a) and (b).

Câmara-Partington ($\beta \leq \alpha$), Ł.-Jurasik, 2016

(a) For w in $\mathbb{D}$, the operators $k_w^\beta \otimes \widetilde{k}_w^\alpha$ and $\widetilde{k}_w^\beta \otimes k_w^\alpha$ belong to $\mathscr{T}(\alpha, \beta)$.

(b) If both α and β have an ADC at the point w of $\partial\mathbb{D}$, then the operator $k_w^\beta \otimes k_w^\alpha$ belongs to $\mathscr{T}(\alpha, \beta)$.

Are the nonzero scalar multiples of the operators in (a) and (b) the only rank-one operators in $\mathscr{T}(\alpha, \beta)$?

Recall that in (b):

$$\widetilde{k}_w^\alpha = \alpha(w)\overline{w}k_w^\alpha$$

and so

$$k_w^\beta \otimes k_w^\alpha = \alpha(w)\overline{w}(k_w^\beta \otimes \widetilde{k}_w^\alpha) = \overline{\beta(w)}w(\widetilde{k}_w^\beta \otimes k_w^\alpha).$$

Are the nonzero scalar multiples of the operators $k_w^\beta \otimes \widetilde{k}_w^\alpha$ and $\widetilde{k}_w^\beta \otimes k_w^\alpha$, $w \in \overline{\mathbb{D}}$, the only rank-one operators in $\mathscr{T}(\alpha, \beta)$?

The trivial case: $\dim K_\alpha = \dim K_\beta = 1$

Let α and β be two inner functions such that

$$\dim K_\alpha = \dim K_\beta = 1.$$

Then

$$\begin{aligned} f \in K_\alpha \quad &\Rightarrow \quad f = c_1 k_0^\alpha, \\ g \in K_\beta \quad &\Rightarrow \quad g = c_2 \widetilde{k}_0^\beta, \end{aligned}$$

and

$$g \otimes f = c(k_0^\beta \otimes \widetilde{k}_0^\alpha).$$

So here the answer is yes.

This is not always the case.

A counterexample

Let $a \in \mathbb{D} \setminus \{0\}$ and let

$$\alpha(z) = z\frac{z-a}{1-\bar{a}z}\frac{z+a}{1+\bar{a}z}, \qquad \beta(z) = z.$$

Then

$$\begin{array}{rcl} K_\alpha & = & \mathrm{span}\{1, k_a, k_{-a}\}, \\ K_\beta & = & \mathcal{P}_0 = \{\lambda \colon \lambda \in \mathbb{C}\} \end{array}$$

(note that $k_w^\beta = \widetilde{k}_w^\beta = 1$).

Since $\dim K_\beta = 1$, every linear operator from K_α into K_β is of rank one.

Put $\varphi = \overline{1 + k_a} \in \overline{K_\alpha^\infty}$. Then

$$A_\varphi^{\alpha,\beta} = 1 \otimes (1 + k_a).$$

Indeed, for every $f \in K_\alpha$, $z \in \mathbb{D}$,

$$\begin{aligned} A_\varphi^{\alpha,\beta} f(z) &= \langle P_\beta(\varphi f), k_z^\beta\rangle = \langle \varphi f, k_z^\beta\rangle \\ &= \langle \overline{(1+k_a)} f, 1\rangle = \langle f, 1 + k_a\rangle = (1 \otimes (1 + k_a))(f). \end{aligned}$$

A counterexample

If

$$A_\varphi^{\alpha,\beta} = 1 \otimes (1 + k_a) = c(\widetilde{k}_w^\beta \otimes k_w^\alpha),$$

for some $w \in \overline{\mathbb{D}}$, then (since $\widetilde{k}_w^\beta = 1$)

$$1 + k_a = \overline{c} k_w^\alpha.$$

Equivalently,

$$\begin{cases} \langle 1 + k_a, 1 \rangle & = & \langle \overline{c} k_w^\alpha, 1 \rangle & \Rightarrow & c = 2 \\ \langle 1 + k_a, k_a \rangle & = & \langle \overline{c} k_w^\alpha, k_a \rangle & \Rightarrow & w = \frac{a}{2 - |a|^2} \\ \langle 1 + k_a, k_{-a} \rangle & = & \langle \overline{c} k_w^\alpha, k_{-a} \rangle & \Rightarrow & w = \frac{a}{2 + |a|^2} \end{cases}.$$

This means that $1 + k_a$ is not a scalar multiple of a reproducing kernel and

$$A_\varphi^{\alpha,\beta} = 1 \otimes (1 + k_a) \neq c(\widetilde{k}_w^\beta \otimes k_w^\alpha).$$

Similarly, $1 + k_a$ is not a scalar multiple of a conjugate kernel and

$$A_\varphi^{\alpha,\beta} = 1 \otimes (1 + k_a) \neq c(k_w^\beta \otimes \widetilde{k}_w^\alpha).$$

Note that if dim $K_\alpha = 1$ or dim $K_\beta = 1$, then every bounded linear operator from K_α into K_β

(a) is of rank one,

(b) is an asymmetric truncated Toeplitz operator.

Proof of (b):

Câmara-Jurasik-Kliś–Garlicka-Ptak ($\beta \leq \alpha$), Gu-Ł.-Michalska, 2017

Let A be a bounded linear operator from K_α into K_β. Then $A \in \mathscr{T}(\alpha, \beta)$ if and only if there exist $\psi \in K_\beta$ and $\chi \in K_\alpha$ such that

$$A - S_\beta A S_\alpha^* = \psi \otimes k_0^\alpha + k_0^\beta \otimes \chi$$

If dim $K_\alpha = 1$ or dim $K_\beta = 1$, then $A - S_\beta A S_\alpha^* = g \otimes f$.

If dim $K_\alpha = 1$, then $f = ck_0^\alpha$ ($\psi = \bar{c}g$, $\chi = 0$).

If dim $K_\beta = 1$, then $g = ck_0^\beta$ ($\psi = 0$, $\chi = \bar{c}f$). □

$\dim K_\alpha > 1$ and $\dim K_\beta = 1$

Let α and β be two inner functions such that $\dim K_\alpha > 1$ and $\dim K_\beta = 1$.

> Every rank-one operator from $\mathscr{T}(\alpha, \beta)$ is a nonzero scalar multiple of $k_w^\beta \otimes \widetilde{k}_w^\alpha$ or $\widetilde{k}_w^\beta \otimes k_w^\alpha$ for some $w \in \overline{\mathbb{D}}$.
>
> $\Updownarrow$
>
> Every function from K_α is a scalar multiple of a reproducing kernel or a conjugate kernel.

Proof: Let $f \in K_\alpha$ and $g \in K_\beta$.

$$k_0^\beta \otimes f \in \mathscr{T}(\alpha, \beta)$$
$$\Downarrow$$
$$k_0^\beta \otimes f = c(k_w^\beta \otimes \widetilde{k}_w^\alpha) \text{ or } k_0^\beta \otimes f = c(\widetilde{k}_w^\beta \otimes k_w^\alpha)$$
$$\Downarrow$$
$$f = c\widetilde{k}_w^\alpha \text{ or } f = ck_w^\alpha$$

$$g \otimes f = c(k_w^\beta \otimes \widetilde{k}_w^\alpha) \text{ or } g \otimes f = c(\widetilde{k}_w^\beta \otimes k_w^\alpha)$$
$$\Uparrow$$
$$f = c\widetilde{k}_w^\alpha \text{ or } f = ck_w^\alpha$$
$$\Uparrow$$
$$g \otimes f \in \mathscr{T}(\alpha, \beta)$$

□

When is every function in the model space a scalar multiple of a reproducing kernel or a conjugate kernel?

Proposition

Every $f \in K_\alpha$ is a scalar multiple of a reproducing kernel or a conjugate kernel if and only if $\dim K_\alpha \leq 2$.

$\dim K_\alpha = 1$ or $\dim K_\beta = 1$

Corollary ($\dim K_\alpha > 1$ and $\dim K_\beta = 1$)

(a) *If* $\dim K_\alpha \leq 2$ *and* $\dim K_\beta = 1$*, then every (rank-one) operator from* $\mathscr{T}(\alpha,\beta)$ *is a scalar multiple of* $k_w^\beta \otimes \widetilde{k}_w^\alpha$ *or* $\widetilde{k}_w^\beta \otimes k_w^\alpha$ *for some* $w \in \overline{\mathbb{D}}$*.*

(b) *If* $\dim K_\alpha > 2$ *and* $\dim K_\beta = 1$*, then there exists a rank-one operator from* $\mathscr{T}(\alpha,\beta)$ *that is neither a scalar multiple of* $k_w^\beta \otimes \widetilde{k}_w^\alpha$ *nor a scalar multiple of* $\widetilde{k}_w^\beta \otimes k_w^\alpha$*.*

Corollary ($\dim K_\alpha = 1$ and $\dim K_\beta > 1$)

(a) *If* $\dim K_\alpha = 1$ *and* $\dim K_\beta \leq 2$*, then every (rank-one) operator from* $\mathscr{T}(\alpha,\beta)$ *is a scalar multiple of* $k_w^\beta \otimes \widetilde{k}_w^\alpha$ *or* $\widetilde{k}_w^\beta \otimes k_w^\alpha$ *for some* $w \in \overline{\mathbb{D}}$*.*

(b) *If* $\dim K_\alpha = 1$ *and* $\dim K_\beta > 2$*, then there exists a rank-one operator from* $\mathscr{T}(\alpha,\beta)$ *that is neither a scalar multiple of* $k_w^\beta \otimes \widetilde{k}_w^\alpha$ *nor a scalar multiple of* $\widetilde{k}_w^\beta \otimes k_w^\alpha$*.*

$\dim K_\alpha > 1$ and $\dim K_\beta > 1$

Theorem

Let α and β be two inner functions such that $\dim K_\alpha > 1$ and $\dim K_\beta > 1$. Then the only rank-one operators in $\mathscr{T}(\alpha, \beta)$ are the nonzero scalar multiples of the operators $k_w^\beta \otimes \widetilde{k}_w^\alpha$ and $\widetilde{k}_w^\beta \otimes k_w^\alpha$ where $w \in \mathbb{D}$ or $w \in \partial\mathbb{D}$ and α and β have an ADC at w.

The proof uses the following lemma.

Lemma

Let α and β be two inner functions such that $\dim K_\alpha > 1$ and $\dim K_\beta > 1$. Let $f \in K_\alpha$, $g \in K_\beta$ be two nonzero functions such that $g \otimes f$ belongs to $\mathscr{T}(\alpha, \beta)$ and let $w \in \overline{\mathbb{D}}$. Then

(a) *g is a scalar multiple of k_w^β if and only if f is a scalar multiple of $\widetilde{k}_w^\alpha$,*

(b) *g is a scalar multiple of $\widetilde{k}_w^\beta$ if and only if f is a scalar multiple of k_w^α.*

Are the nonzero scalar multiples of the operators $k_w^\beta \otimes \widetilde{k}_w^\alpha$ and $\widetilde{k}_w^\beta \otimes k_w^\alpha$, $w \in \overline{\mathbb{D}}$, the only rank-one operators in $\mathscr{T}(\alpha, \beta)$?

Theorem

Let α and β be two inner functions such that $\dim K_\alpha = m$ and $\dim K_\beta = n$ (with m or n possibly infinite). The only rank-one operators in $\mathscr{T}(\alpha, \beta)$ are the nonzero scalar multiples of the operators $k_w^\beta \otimes \widetilde{k}_w^\alpha$ and $\widetilde{k}_w^\beta \otimes k_w^\alpha$, $w \in \overline{\mathbb{D}}$, if and only if either $mn \leq 2$, or $m > 1$ and $n > 1$.

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Thank you for your attention!