Dimension free bounds for the vector-valued Hardy-Littlewood maximal operator

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The Hardy-Littlewood maximal function

Let $n \in \mathbb{N}$. Consider a function $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Define

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy \in [0, \infty].$$

M is a sublinear operator: $|M(cf)| = |c| \, |Mf|$, $|M(f+g)| \leq |Mf| + |Mg|$, $|Mf - Mg| \leq |M(f-g)|$.
M is called the Hardy-Littlewood maximal operator.

Boundedness properties of the Hardy-Littlewood maximal function

Theorem (Hardy-Littlewood 1930):

1. There exists a constant $C(n) < \infty$ such that
$$|\{x : (Mf)(x) > \alpha\}| \leq \frac{C(n)}{\alpha} \int_{\mathbb{R}^n} |f(y)| dy \quad (\alpha > 0).$$
"weak type $(1,1)$ estimate".
2. Let $1 < p \leq \infty$. There exists a constant $C(n,p) < \infty$ such that
$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq C(n,p)\|f\|_{L^p(\mathbb{R}^n)}.$$
"strong type (p,p) estimate".

Proof of Hardy's and Littlewood's theorem

Part 1: Fix some $\alpha > 0$. Let $\{x : (Mf)(x) > \alpha\} =: E_\alpha$. Let $E \subset E_\alpha$ be compact. For each $x \in E$, there is $B(x, r_x)$ such that

$$|B(x, r_x)| \leq \frac{1}{\alpha} \int_{B(x,r_x)} |f(y)| dy.$$

$\bigcup_{x \in E} B(x, r_x) \supseteq E$, so by compactness $\bigcup_{k=1}^{K} B(x_k, r_{x_k}) \supseteq E$.
Vitali Covering Lemma: Out of $\{B(x_1, r_{x_1}), \ldots, B(x_K, r_{x_K})\}$ we can choose a *pairwise disjoint* subcollection $\{B_1, \ldots, B_m\}$ such that $|E| \leq C(n) \sum_{k=1}^{m} |B_k|$.
Thus

$$|E| \leq \frac{C(n)}{\alpha} \sum_{k=1}^{m} \int_{B_k} |f(y)| dy \leq \frac{C(n)}{\alpha} \int_{\mathbb{R}^n} |f(y)| dy.$$

But $E \subset E_\alpha$ was arbitrary.

Proof of Hardy's and Littlewood's theorem

Part 2: On the level $p = \infty$, we have
$|Mf(x)| = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy \leq \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \|f\|_\infty dy = \|f\|_\infty$.

But M is sublinear.
Now apply Marcinkiewicz real interpolation between the cases weak $(1,1)$ and strong (∞, ∞) to deduce Part 2. QED.

Application and related questions

Corollary: Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then $\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy$ converges to $f(x)$ for a.e. $x \in \mathbb{R}^n$ as $r \to 0$.
Other classical applications of the boundedness of M: boundedness of singular integral operators.
Related questions: Replace the ball in the definition of M by (dilates of) cube or another symmetric convex body. Boundedness of the resulting maximal operator ? Works in the period 1986 - 1990 by Bourgain, Carbery, Müller, ... and in 2010's by Bourgain.

Question: What are the optimal bounds $C(n)$ and $C(n,p)$ in Hardy-Littlewood's theorem?
We have seen: $C(n,\infty) = 1$.
Melas 2003: If $n = 1$ then $C(1) = \frac{11+\sqrt{61}}{12}$.
Other values of best constants: ??? Unknown today.

Dimension free bound of Hardy-Littlewood maximal operator

Next Question: Can one bound $C(n) \leq C$ or $C(n,p) \leq C(p)$ for any $n \in \mathbb{N}$?
Theorem Stein-Strömberg 1983: $C(n,p) \leq C(p)$ for $1 < p < \infty$.
Part of Proof: Introduce the *spherical maximal operator*

$$M_S f(x) = \sup_{r>0} \frac{1}{|S(x,r)|} \left| \int_{S(x,r)} f(y)dy \right|.$$

Here, $S(x,r)$ is the sphere of radius r and $|S(x,r)|$ denotes its (surface) measure. Show that M_S is bounded on $L^p(\mathbb{R}^n)$ for $n \geq 3$ and $p > \frac{n}{n-1}$.
The integrals in the def. of M are averages of the integrals in the def. of M_S. Thus, $Mf(x) \leq M_S f(x)$. Still $\|M_S\| = B(n,p)$, but we can do better:

The spherical maximal operator

Introduce the auxiliary maximal operator

$$M_{n'}^{\theta} f(x) = \sup_{r>0} \frac{\int_{|y_{n'}|\le r} |f(x-\theta(y_{n'},0))|\, |y_{n'}|^{n-n'} dy_{n'}}{\int_{|y_{n'}|\le r} |y_{n'}|^{n-n'} dy_{n'}} \quad (x \in \mathbb{R}^n),$$

where θ is a rotation matrix over $\mathbb{R}^n$ and $n' \le n$ a lower dimension. Show that $\|M_{n'}^{\theta}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \le A(n', p)$ **using** $M_S : L^p(\mathbb{R}^{n'}) \to L^p(\mathbb{R}^{n'})$. Moreover, show that $Mf(x) \le \int_{O(n)} M_{n'}^{\theta} f(x) d\mu(\theta)$.
Now fix some small n'. $\|M\|_{p\to p} \le \int_{O(n)} \|M_{n'}^{\theta}\|_{p\to p} \le A(n', p)$ as soon as spherical maximal operator M_S is available, i.e. $n' \ge 3$ and $p > \frac{n'}{n'-1}$. This is the case for $p > 1$ fixed from some fixed initial dimension n' on. For $n \ge n'$ we deduce the dimension free bound of $\|M\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}$ and for the remaining small n, we have the classical Hardy-Littlewood theorem. QED.

Vector valued Hardy-Littlewood maximal operator

Remark: Stein-Strömberg: $C(n,p) \leq C(p)$. But in the weak-$(1,1)$ estimate, $C(n) \leq C$: ??? Unkown today, probably false.
Now consider a second Lebesgue exponent $q \in [1,\infty]$.
Can ask the question

$$\left\| \left(\sum_{k=1}^{\infty} |Mf_k|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \leq C(n,p,q) \left\| \left(\sum_{k=1}^{\infty} |f_k|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \; ?$$

These are the Fefferman-Stein inequalities.
Theorem Fefferman-Stein (1971): Yes for $1 < p, q < \infty$.
Theorem Grafakos-Liu-Yang (2009): Yes for
$1 < p < \infty$, $1 < q \leq \infty$ if one replaces $\mathbb{R}^n$ by a metric measure space of homogeneous type.

Dimensionless Fefferman-Stein inequalities ?

Again question: Can one bound $C(n, p, q) \leq C(p, q)$ for all $n \in \mathbb{N}$?
Proof of Fefferman-Stein: Uses Calderon-Zygmund decomposition related to the proof of Hardy-Littlewood theorem. Vitali covering lemma: $C(n, p, q) \leq 3^n C(p, q)$.
Other proofs of Fefferman-Stein inequalities by Garcia-Cuerva, Rubio de Francia and Grafakos also do not yield dimension free bounds.

Dimensionless Fefferman-Stein inequalities

Theorem (Deléaval-K. 2016): We have a dimension-free bound

$$\left\| \left(\sum_{k=1}^{\infty} |Mf_k|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \leq C(p,q) \left\| \left(\sum_{k=1}^{\infty} |f_k|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}$$

for $1 < p, q < \infty$.

Ideas of the proof: Use Stein-Strömberg's approach of spherical maximal operator.

Thus question: Is M_S bounded $L^p(\mathbb{R}^n; \ell^q) \to L^p(\mathbb{R}^n; \ell^q)$?

We have $M_S f(x) = \sup_{r>0} |A_r f(x)|$ with $A_r f(x) = \frac{1}{|S(0,r)|} \int_{S(0,r)} f(x-y)dy$. A_r is a convolution operator, therefore a Fourier multiplier operator. One can compute its symbol! $A_r f(x) = (m(r\cdot)\hat{f})\check{}(x)$ with $m(x) = \frac{2\pi}{|x|^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(2\pi|x|)$, where $J_{\frac{d-2}{2}}$ is a Bessel function.

Proof of dimensionless Fefferman-Stein

So M_S is a maximal operator associated to a (radial) Fourier multiplier. When are Fourier multipliers bounded on $L^p(\mathbb{R}^n)$?
Sufficient criterion: When its Hörmander norm is finite:
$\|m\|_H^2 = \max_{k=0}^{\alpha} \sup_{r>0} \int_r^{2r} |t^k m^{(k)}(t)|^2 \frac{dt}{t} < \infty$ for $\alpha > n/2$.
We need more than Fourier multiplier: *maximal* Fourier multiplier, and on vector-valued $L^p(\mathbb{R}^n; \ell^q)$.

Theorem (Deléaval-K. 2016): The maximal radial Fourier multiplier $f \mapsto \sup_{r>0} \left| (m(r\cdot)\hat{f})^\vee \right|$ is bounded on $L^p(\mathbb{R}^n; \ell^q)$ provided that $\|m\|_{H\max}^2 = \max_{k=0}^{\alpha} \sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} |t^k m^{(k)}(t)|^2 \frac{dt}{t} < \infty$ for $\alpha > n(\frac{1}{2} - \epsilon_{p,q}) + 2$.
Now calculate with the Bessel function: $\|m\|_{H\max} < \infty$ if $\alpha \leq \frac{n}{2} - \frac{3}{2}$.
Conclusion: M_S is bounded on $L^p(\mathbb{R}^n; \ell^q)$ provided that $n > \frac{1}{\epsilon_{p,q}}(2 + \frac{3}{2})$.

Conclusion of the proof

For big dimensions $n > \frac{1}{\epsilon_{p,q}}(2 + \frac{3}{2})$, we can use the rotated maximal operator of lower dimension and conclude as in the proof of Stein's and Strömberg's result. For small dimensions, we can invoke the classical Fefferman-Stein inequalities. QED.

Extensions of the dimensionless theorem

In the dimensionless Fefferman-Stein theorem, one can replace ℓ^q by any UMD lattice:
Indeed, the maximal Fourier multiplier theorem still holds. For small dimensions, there is now a different argument from [Xu 2015 IRMN].

One can also ask to replace $\mathbb{R}^n$ by .... e.g. other Lie groups of polynomial growth,
or the Laplacian underlying the Fourier multiplier symbol by another operator.
We have some results for the so-called Grushin operator on $\mathbb{R}^n \times \mathbb{R}$, and a maximal multiplier theorem for the sub-Laplacian on the Heisenberg group (in fact, abstract max. mult. theorem).

Thank you for your attention