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On the symmetrization of general Wiener-Hopf operators

Abstract

This article focuses on general Wiener-Hopf operators given as $W = P_2 A|_{P_1 X}$ where X, Y are Banach spaces, $P_1 \in \mathcal{L}(X)$, $P_2 \in \mathcal{L}(Y)$ are any projectors and $A \in \mathcal{L}(X, Y)$ is boundedly invertible. It presents conditions for W to be equivalently reducible to a Wiener-Hopf operator in a symmetric space setting where $X = Y$ and $P_1 = P_2$. The results and methods are related to the so-called Wiener-Hopf factorization through an intermediate space and the construction of generalized inverses of W in terms of factorizations of A.

The talk is based upon joint work with Albrecht Böttcher, in J. Operator Theory 2016.

General Wiener-Hopf operators

Let X, Y be Banach spaces, $A \in \mathcal{L}(X,Y)$, $P_1 \in \mathcal{L}(X)$, $P_2 \in \mathcal{L}(Y)$ projectors, $Q_1 = I_X - P_1$, $Q_2 = I_Y - P_2$.

Then the operator

$$W \quad = \quad P_2 A|_{P_1 X} \quad = \quad P_1 X \to P_2 Y \tag{1}$$

is referred to as a general Wiener-Hopf operator (WHO). We assume that the so-called underlying operator A is invertible, i.e., that A is a linear homeomorphism, written as $A \in G\mathcal{L}(X,Y)$. In a sense, this is no limitation of generality; see, e.g., S14.

In a symmetric setting, where $X = Y, P_1 = P_2 = P$, the operator W is commonly written in the form (see Shi64, DevShi69)

$$W = T_P(A) = PA|_{PX} : PX \to PX \tag{2}$$

and also called an abstract Wiener-Hopf operator Ceb67 or a projection or a truncation or a compression of A GohKru79.

Questions

Question 1 When is the operator W in (1) equivalent to a WHO $\tilde{W}$ in symmetric setting (2)? I.e. there exists a space Z, an operator $\tilde{A} \in G\mathcal{L}(Z)$, a projector $P \in \mathcal{L}(Z)$ and isomorphisms E, F such that

$$W \quad = \quad P_2 A|_{P_1 X} \quad = \quad E \; \tilde{W} \; F \quad = \quad E \, P \tilde{A}|_{PZ} \, F \, .$$

The answer depends heavily on all "parameters" X, Y, P_1, P_2, A and is particularly trivial for finite rank operators W or for separable Hilbert spaces X, Y. Hence we modify the question:

Question 2 When is the operator W of (1) equivalent to a WHO $\tilde{W}$ in symmetric setting (2), for any choice of $A \in G\mathcal{L}(X, Y)$?

Remark.

This does not imply that E and F are independent of A, but has to do with factorizations of A. The answer can be seen as a property of the space setting X, Y, $\operatorname{im} P_1, \ker P_2$, as we shall see.

Motivation

A strong motivation to study the operator (1) in an asymmetric space setting is given by the theory of pseudo-differential operators, which naturally act between Sobolev-like spaces of different orders; see Eskin's book 1973/81. Their symmetrization (lifting) by generalized Bessel potential operators is considered in DudSpe93.

Furthermore, Toeplitz operators with singular symbols are another source of motivation for considering symmetrization. We will briefly touch these two concrete applications in the examples later on.

Idea of the paper

In 1985, the second author introduced the notion of a cross factorization and proved that the generalized invertibility of W is equivalent to the existence of a cross factorization of A.

In a recent paper S14, two further kinds of operator factorizations were studied, the Wiener-Hopf factorization of A through an intermediate space and the full range factorization $W = LR$ where L is left invertible and R is right invertible. The main theorem of S14 states the equivalence between all three factorizations, partly under the restrictive condition that the two projectors P_1 and P_2 are equivalent.

Unfortunately, one proof in S14 contains a gap. This gap, which was filled in of the present paper, actually motivated us to look after the matter again. Our efforts resulted in a symmetrization criterion (Theorem 1 below) and a new proof of a basic theorem of S14 (Theorem 2 below).

Symmetrizable space settings

Our first topic here is the symmetrization of asymmetric WHOs.

To be more precise, we call the setting X, Y, P_1, P_2 symmetrizable if there exist a Banach space Z, operators $M_+ \in G\mathcal{L}(X, Z)$ and $M_- \in G\mathcal{L}(Z, Y)$, and a projector $P \in \mathcal{L}(Z)$ such that

$$M_+(P_1 X) = PZ, \quad M_-(QZ) = Q_2 Y, \tag{3}$$

where $Q = I_Z - P$ and $Q_2 = I_Y - P_2$.

Note that the invertibility of M_+ and M_- in conjunction with (3) implies that

$$U_+ := M_+|_{P_1 X} : P_1 X \to PZ, \quad V_- := M_-|_{QZ} : QZ \to Q_2 Y, \tag{4}$$

are invertible.

Symmetrization of asymmetric WHOs

If the setting X, Y, P_1, P_2 is symmetrizable, then asymmetric WHOs may also be symmetrized: given an operator of the form (1), there is an operator $\widetilde{A} \in \mathcal{L}(Z)$ such that $A = M_- \widetilde{A} M_+$ and $W = V_+ \widetilde{W} U_+ = V_+ T_P(\widetilde{A}) U_+$. Indeed, we have $\widetilde{A} = M_-^{-1} A M_+^{-1}$, and since $PM_-^{-1} = PM_-^{-1} P_2$ and $PM_+ P_1 = M_+ P_1$, we get

$$\begin{aligned} V_+ \widetilde{W} U_+ &= (PM_-^{-1}|_{P_2 Y})^{-1} \, PM_-^{-1} A M_+^{-1}|_{PZ} \, (PM_+|_{P_1 X}) \\ &= (PM_-^{-1}|_{P_2 Y})^{-1} \, PM_-^{-1} P_2 A M_+^{-1} \, M_+|_{P_1 X} \\ &= P_2 A|_{P_1 X} = W \, . \end{aligned}$$

As usual, we call two operators T and S equivalent, written $T \sim S$, if there exist linear homeomorphisms E and F such that $T = FSE$.

Thus, in the case of a symmetrizable setting,

$$W \sim T_P(\widetilde{A}).$$

Main result

Given two Banach spaces Z_1 and Z_2, we write $Z_1 \cong Z_2$ if the two spaces are isomorphic, that is, if there exists an operator A in $G\mathcal{L}(Z_1, Z_2)$. We also put $Q_1 = I_X - P_1$, $Q_2 = I_Y - P_2$.

Theorem 1 *The following are equivalent:*

(i) *the setting X, Y, P_1, P_2 is symmetrizable,*
(ii) $P_1 X \cong P_2 Y$ *and* $Q_1 X \cong Q_2 Y$,
(iii) $P_1 \sim P_2$.

The theorem implies in particular that every setting given by two separable Hilbert spaces X, Y and two infinite-dimensional bounded projectors P_1, P_2 with isomorphic kernels is symmetrizable. Many examples from applications satisfy this condition.

Related results

Later on we shall recall two types of factorizations of the underlying operator A, the cross factorization (CFn) and the Wiener-Hopf factorization through an intermediate space (FIS). Note that the existence of a CFn for A is equivalent to the generalized invertiblity of W in the sense that there exists an operator $W^- \in \mathcal{L}(P_2Y, P_1X)$ such that $WW^-W = W$. Herewith our second main result:

Theorem 2 *Given a setting X, Y, P_1, P_2. The following assertions are equivalent:*

(i) *A has a CFn and $P_1 \sim P_2$,*
(ii) *A has a FIS.*

Theorem 2 is already in S14, and it is the theorem whose proof in that paper contains a gap. We here give another, more straightforward proof. In addition we repair the gap of the proof in S14, thus saving also the original proof.

Remark

From Theorem 1 we see that if $P_1 \sim P_2$, then

$$P_1X \times Q_2Y \cong P_1X \times Q_1X \cong P_1X \oplus Q_1X = X,$$
$$P_1X \times Q_2Y \cong P_2Y \times Q_2Y \cong P_2Y \oplus Q_2Y = Y,$$

and hence

$$X \cong P_1X \times Q_2Y \cong Y. \tag{5}$$

However, (5) does not imply that $P_1 \sim P_2$. A counterexample is provided by the setting $X = Y = \ell^2(\mathbb{Z})$,

$$P_1 : (..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...) \mapsto (..., 0, 0, 0, x_1, x_2, ...),$$
$$P_2 : (..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...) \mapsto (..., 0, 0, x_0, 0, 0, ...).$$

Condition (5) holds because X, Y, P_1X, Q_2Y are infinite-dimensional separable Hilbert spaces, but P_1 and P_2 are clearly not equivalent.

Example 1: Toeplitz operators with FH symbols

A concrete case where symmetrization was used (without calling it symmetrization) occurs in the proof of the Fisher-Hartwig conjecture in BS85. A Fisher-Hartwig symbol is a function of the form

$$a(t) = b(t) \prod_{j=1}^{N} |t - t_j|^{2\alpha_j}, \quad t \in \mathbb{T},$$

where b is a piecewise continuous function on $\mathbb{T}$ that is invertible in L^∞, $t_1, \ldots, t_N$ are distinct points on $\mathbb{T}$, and $\alpha_1, \ldots, \alpha_N$ are complex numbers whose real parts lie in the interval $(-1/2, 1/2)$. The Toeplitz operator generated by a is an operator of the form $T(a) = P_2 M(a)|_{\operatorname{im} P_1}$, where $M(a)$ acts on certain Lebesgue spaces over $\mathbb{T}$ by the rule $f \mapsto af$ and P_1, P_2 are the Riesz projectors of the Lebesgue spaces onto their Hardy spaces. The operators $M(a)$ and $T(a)$ are in general neither bounded nor invertible on L^p and the corresponding Hardy spaces H^p. However, things can be saved by passing to weighted spaces. Put $\varrho(t) = \prod_{j=1}^{N} |t - t_j|^{\mathrm{Re}\,\alpha_j}$.

For $1 < p < \infty$, let

$$L^p(\varrho^{\pm 1}) = \left\{ f \in L^1 : \|f\|^p := \int_{\mathbb{T}} |f(t)|^p \varrho(t)^{\pm p} |dt| < \infty \right\}.$$

The Riesz projector P, which may be defined as $P = (I + S)/2$ with the Cauchy singular integral operator S given by

$$(Sf)(t) = \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{|\tau - t| > \varepsilon} \frac{f(\tau)}{\tau - t} \, d\tau, \quad t \in \mathbb{T},$$

is bounded on the spaces $L^p(\varrho^{\pm 1})$ if $\mathrm{Re}\alpha_j \in (-1/r, 1/r)$ where $r = \max(p, q)$ with $1/p + 1/q = 1$. Thus, assume the real parts $\mathrm{Re}\alpha_j$ are all in $(-1/r, 1/r)$.

Finally, consider the setting

$$X = L^p(\varrho), \quad P_1 = P, \quad Y = L^p(\varrho^{-1}), \quad P_2 = P.$$

It turns out that $M(a) \in G\mathcal{L}(X, Y)$ and hence we are in the setting (1) with the invertible operator $A = M(a)$. The Toeplitz operator $T(a)$ acts from $PL^p(\varrho)$ to $PL^p(\varrho^{-1})$. Thus, it is a WHO in an asymmetric setting. It can be shown that the setting X, Y, P_1, P_2 is symmetrized by $Z = L^p$, $P =$ Riesz projector, $M_+ := M(\eta)$, $M_- := M(\xi)$, where

$$\eta(t) = \prod_{j=1}^{N} (1 - t/t_j)^{\alpha_j}, \quad \xi(t) = \prod_{j=1}^{N} (1 - t_j/t)^{\alpha_j}.$$

We have $T(a) = V_+ T(b) U_+$ with $T(b) \in \mathcal{L}(L^p, L^p)$, which reduces the study of $T(a)$ to the investigation of the much simpler operator $T(b)$.

Example 2: Lifting of WHOs in Sobolev-like spaces

Another useful application of symmetrization is the reduction of WHOs and pseudo-differential operators in scales of Sobolev spaces to operators acting in L^p spaces by Bessel potential operators for a half-line, half-space, quarter plane, or Lipschitz domain DS93, Esk81, MoST98.

The same idea works for Wiener-Hopf plus/minus Hankel operators, convolution type operators with symmetry, and convolutionally equivalent operators CS15, and it also works for other scales of spaces such as the Sobolev-Slobodetski spaces $W^{s,p}$ and the Zygmund spaces Z^s, as well as for matrix operators, cf. CDS06. To illustrate the strategy, we here confine us to the basic variant of classical WHOs in Bessel potential spaces (one-dimensional, scalar, $p = 2$).

Let $\mathcal{F}$ be the Fourier transformation, $(\mathcal{F}f)(\xi) = \int_{\mathbb{R}} f(x)e^{i\xi x}dx$, and let H^s denote the Sobolev space of all distributions f on $\mathbb{R}$ such that $\lambda^s \mathcal{F}f \in L^2$, where $\lambda(\xi) = (\xi^2+1)^{1/2}$. The well-known Bessel potential operators are given by

$$\Lambda^s := A_{\lambda^s} := \mathcal{F}^{-1}\lambda^s \cdot \mathcal{F} : H^r \to H^{r-s},$$
$$\Lambda^s_\pm := A_{\lambda^s_\pm} := \mathcal{F}^{-1}\lambda^s_\pm \cdot \mathcal{F} : H^r \to H^{r-s},$$

where $\lambda_\pm(\xi) = \xi \pm i$; see, for example, Dud79, Esk81, MoST98. Here r and s are real numbers.

Let H^s_+ and H^s_- stand for the subspace of all distributions in H^s that are supported on $[0, \infty)$ and $(-\infty, 0]$, respectively. We then have

$$\Lambda^s_+(H^r_+) = H^{r-s}_+, \quad \Lambda^s_-(H^r_-) = H^{r-s}_-.$$

In terms of operator identities, this may be rephrased as follows. If $P_1^{(s)}$ and $P_2^{(s)}$ are any bounded projectors on H^s such that $\operatorname{im} P_1^{(s)} = H^s_+$ and $\ker P_2^{(s)} = H^s_-$, then

$$\Lambda^s_+ P_1^{(r)} = P_1^{(r-s)} \Lambda^s_+ P_1^{(r)}, \quad P_2^{(r-s)} \Lambda^s_- = P_2^{(r-s)} \Lambda^s_- P_2^{(r)}.$$

In accordance with Esk81, a classical Wiener-Hopf operator is given by

$$T = r_+ \, A_\Phi|_{H^r_+} : H^r_+ \to H^s(\mathbb{R}_+)$$

where $H^s(\mathbb{R}_+)$ is the common Hilbert space of all restrictions of distributions in H^s to $\mathbb{R}_+ = (0, \infty)$, $r_+ : f \mapsto f|_{\mathbb{R}_+}$ is the restriction operator, and A_Φ is a convolution (or translation invariant) operator of order $r - s$, that is, A_Φ is of the form

$$A_\Phi = \mathcal{F}^{-1} \Phi \cdot \mathcal{F} : H^r \to H^s \quad \text{with} \quad \lambda^{s-r} \Phi \in L^\infty(\mathbb{R}).$$

Obviously, T is equivalent to the general Wiener-Hopf operator W given by

$$W = P_2^{(s)} A_\Phi|_{H_+^r} : P_1^{(r)} H^r \to P_2^{(s)} H^s,$$

where $P_2^{(s)} := \ell^{(s)} r_+ \in \mathcal{L}(H^s)$ and $\ell^{(s)} : H^s(\mathbb{R}_+) \to H^s$ is any bounded extension operator that is left invertible by r_+. The projector $P_1^{(r)}$ may be an arbitrary projector in $\mathcal{L}(H^r)$ such that $\operatorname{im} P_1^{(r)} = H_+^r$.

The equivalence between T and W is simply given by $W = \ell^{(s)} T$ and $T = r_+ W$. Thus, in the case at hand the setting X, Y, P_1, P_2 is $H^r, H^s, P_1^{(r)}, P_2^{(s)}$. As an interpretation of results in MoST98, a symmetrization of W is achieved by the so-called lifting to L^2: choosing

$$Z := H^0 = L^2(\mathbb{R}), \quad M_+ := \Lambda_+^r, \quad M_- := \Lambda_-^{-s}, \quad P := \ell_0 r_+,$$

where $\ell_0 : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ denotes the extension by zero, we get, with $\Phi_0 := \lambda_-^s \Phi \lambda_+^{-r}$, $P_1^{(0)} := \ell_0 r_+$, $P_2^{(0)} := \ell_0 r_+$,

$$
\begin{aligned}
W &= P_2^{(s)} A_\Phi|_{H_+^r} = P_2^{(s)} \Lambda_-^{-s} A_{\Phi_0} \Lambda_+^r|_{H_+^r} \\
&= P_2^{(s)} \Lambda_-^{-s}|_{P_2^{(0)} H^0} \quad P_2^{(0)} A_{\Phi_0}|_{H_+^0} \quad P_1^{(0)} \Lambda_+^r|_{H_+^r} \\
&= P_2^{(s)} \Lambda_-^{-s}|_{L_+^2} \quad P A_{\Phi_0}|_{L_+^2} \quad P \Lambda_+^r|_{H_+^r} =: E \;\; W_0 \;\; F.
\end{aligned}
$$

Related topics: General WH-Factorization

Let X, Y be Banach spaces, let $P_1 \in \mathcal{L}(X), P_2 \in \mathcal{L}(Y)$ be projectors, and let A be an operator in $G\mathcal{L}(X,Y)$. A factorization

$$\begin{array}{rcl} A & = & A_- \quad C \quad A_+ \\ & : & Y \leftarrow Y \leftarrow X \leftarrow X\,. \end{array}$$

is referred to as a cross factorization of A (with respect to X, Y, P_1, P_2) S83, in brief CFn, if the factors $A_\pm$ and C possess the properties

$$A_+ \in G\mathcal{L}(X), \qquad A_- \in G\mathcal{L}(Y), \tag{6}$$

$$A_+(P_1 X) = P_1 X, \qquad A_-(Q_2 Y) = Q_2 Y,$$

and $C \in G\mathcal{L}(X,Y)$ splits the spaces X, Y both into four complemented subspaces such that

$$\begin{array}{ccccc} X & = & \overbrace{X_1 \quad \oplus \quad X_0}^{P_1 X} & \oplus & \overbrace{X_2 \quad \oplus \quad X_3}^{Q_1 X} \\ & & \downarrow \qquad\qquad\qquad & C \searrow\!\!\!\!\!\!\swarrow & \qquad\qquad\qquad \downarrow \\ Y & = & \underbrace{Y_1 \quad \oplus \quad Y_2}_{P_2 Y} & \oplus & \underbrace{Y_0 \quad \oplus \quad Y_3}_{Q_2 Y}. \end{array} \tag{7}$$

The operators $A_\pm$ are called strong WH factors and C is said to be a cross factor, since it maps a part of P_1X onto a part of Q_2Y $(X_0 \to Y_0)$ and a part of Q_1X onto a part of P_2Y $(X_2 \to Y_2)$, which are all complemented subspaces.

The cross factorization theorem S83

Suppose X, Y are Banach spaces, $P_1 \in \mathcal{L}(X), P_2 \in \mathcal{L}(Y)$ are projectors, and A is an operator in $G\mathcal{L}(X, Y)$.

Then W is generalized invertible (i.e. $WW^-W = W$ for some $W^- \in \mathcal{L}(Y, X)$) if and only if a cross factorization of A exists.

In that case a formula for a generalized inverse of W is given by

$$W^- = A_+^{-1} P_1 C^{-1} P_2 A_-^{-1}|_{P_2 Y} : P_2 Y \to P_1 X.$$

A crucial consequence is the equivalence of W and $P_2 C|_{P_1 X}$, that is, $W \sim P_2 C|_{P_1 X}$:

$$W = P_2 A_-|_{P_2 Y} \; P_2 C|_{P_1 X} \; P_1 A_+|_{P_1 X} = E \; P_2 C|_{P_1 X} \; F$$

where E, F are linear homeomorphisms. We refer to S85 for more details.

Remark: The proof (of the necessity part) is much simpler for symmetric settings. Hence: Symmetrization counts!

WH factorization through an intermediate space
CS95, S14

Under the same assumptions as before, a factorization

$$\begin{array}{cccccc} A & = & A_- & C & A_+ & \\ & : & Y \leftarrow Z & \leftarrow Z & \leftarrow X\,. & \end{array}$$

is called a Wiener-Hopf factorization through an intermediate space Z (with respect to the setting X, Y, P_1, P_2), in brief FIS, if Z, $A_\pm$, and C possess the following properties:
(a) Z is a Banach space,
(b) $A_+ \in G\mathcal{L}(X, Z)$, $C \in G\mathcal{L}(Z)$, $A_- \in G\mathcal{L}(Z, Y)$,
(c) there exists a projector $P \in \mathcal{L}(Z)$ such that, with $Q := I_Z - P$,

$$A_+(P_1 X) = PZ, \quad A_-(QZ) = Q_2 Y, \tag{8}$$

(d) C splits the space Z twice into four complemented subspaces such that

$$\begin{array}{ccccc} Z & = & \overbrace{X_1 \quad \oplus \quad X_0}^{PZ} & \oplus & \overbrace{X_2 \quad \oplus \quad X_3}^{QZ} \\ & & \downarrow \qquad\qquad\quad & C \; \times & \qquad\qquad\quad \downarrow \\ Z & = & \underbrace{Y_1 \quad \oplus \quad Y_2}_{PZ} & \oplus & \underbrace{Y_0 \quad \oplus \quad Y_3}_{QZ}. \end{array} \tag{9}$$

Again $A_{\pm}$ are called strong WH factors and C is said to be a cross factor, now acting from a space Z onto the same space Z. If the factor C in a FIS is the identity, we speak of a canonical FIS.

Immediate consequences

Remark 3 *A FIS of A implies the equivalence relation*

$$W \quad = \quad P_2 A_-|_{PZ} \;\; PC|_{PZ} \;\; PA_+|_{P_1X} \quad \sim \quad PC|_{PZ}\,,$$

which represents a symmetrization of the WHO W *defined in (1).*

As in the case of a CFn it implies the representation of a generalized inverse of W*:*

$$W^- = A_+^{-1}\,P\,C^{-1}\,P\,A_-^{-1}|_{P_2Y} \;\;:\;\; P_2Y \to P_1X.$$

Sketch of the proof of Theorem 1

Theorem 1 (recalled) *The following are equivalent:*

(i) *the setting* X, Y, P_1, P_2 *is symmetrizable,*
(ii) $P_1 X \cong P_2 Y$ and $Q_1 X \cong Q_2 Y$,
(iii) $P_1 \sim P_2$.

(i) $\Rightarrow$ (ii) results from the mapping properties of $M_\pm$ (via Z),

(ii) $\Rightarrow$ (iii) an elementary conclusion,

(iii) $\Rightarrow$ (i) is also elementary, but needs a little effort, various possible proofs exist.

Sketch of the proof of Theorem 2

Theorem 2 *Given a setting X, Y, P_1, P_2. The following assertions are equivalent:*

(i) *A has a CFn and $P_1 \sim P_2$,*

(ii) *A has a FIS.*

(i) $\Rightarrow$ (ii)

1. W is symmetrizable, $W \sim \tilde{W} = P\tilde{A}|_{PZ}$;

2. W is generalized invertible (by the cross factorization theorem), $\tilde{W}$ is generalized invertible (by equivalence), $\tilde{A}$ has a CFn in symmetric setting, which represents a FIS.

(ii) $\Rightarrow$ (i)

1. W is generalized invertible (W^- results from a FIS), hence A has a CFn (by the cross factorization theorem);

2. A FIS of $A = A_- C A_+$ through Z implies that the setting X, Y, P_1, P_2 is symmetrized by putting $M_+ = A_+$ and $M_- = A_-$;

3. Theorem 1 implies that $P_1 \sim P_2$.

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