p-ellipticity

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Problem in the calculus of variations

$F : \mathbb{R}^n \to \mathbb{R}$ of class C^∞ and *strongly convex*, i.e., $\exists\ 0 < \lambda < \Lambda$ such that for all $p, \xi \in \mathbb{R}^n$ we have

$$\lambda|\xi|^2 \leqslant \langle d^2 F(p)\xi, \xi\rangle \leqslant \Lambda|\xi|^2$$

$\Omega \subset \mathbb{R}^n$ bounded domain, $\phi \in C^1(\overline{\Omega})$ given.

Variational problem (VP)

Minimize the functional

$$I(v) := \int_\Omega F(\nabla v)\, dm$$

among all $v \in H^1(\Omega)$ with $v\big|_{\partial\Omega} = \phi$ (in the trace sense).

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VP has a unique minimizer. This solves **Hilbert's 20th problem**.

Are the solutions of regular problems in the calculus of variations always necessarily analytic?

D. Hilbert (ICM Paris 1900)

"Eine der begrifflich merkwürdigsten Tatsachen in den Elementen der Theorie der analytischen Funktionen erblicke ich darin, daß es partielle Differentialgleichungen gibt, deren Integrale sämtlich notwendig analytische Funktionen der unabhängigen Variablen sind, die also, kurz gesagt, nur analytischer Lösungen fähig sind."

Euler-Lagrange equation for minimizers

Suppose u minimizes (VP) and $A = \operatorname{Hess} F(\nabla u)$.
Then $\tilde{u} := \partial_{x_k} u$ is in $V \subset\subset \Omega$ a weak solution of

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Remedy:
Regularity theory that relies only on the **ellipticity** of the matrix.

$A = [a_{ij}] : \Omega \to \mathbb{C}^{n,n}$ is said to be a complex **uniformly strictly accretive (or elliptic)** $n \times n$ matrix function on Ω with L^∞ coefficients if $a_{ij} \in L^\infty(\Omega)$ and $\exists\ \lambda > 0$ such that for a.e. $x \in \Omega$,

$$\Re\langle A(x)\xi, \xi\rangle \geqslant \lambda|\xi|^2\,, \qquad\qquad \forall\xi \in \mathbb{C}^n$$

Here $|\xi|^2 = \langle \xi, \xi\rangle_{\mathbb{C}^n}$. Let $\Lambda = \|A\|_\infty$ and $L_A u := -\mathrm{div}\,(A\nabla u)$. Denote the set of all such matrix functions by $\mathcal{A}_{\lambda,\Lambda}(\Omega)$.

Theorem (E. De Giorgi 1957, J. Nash 1958, J. Moser 1960)

Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and $A \in \mathcal{A}_{\lambda,\Lambda}(\Omega)$ is **real** *symmetric. Then every weak solution $v \in H^1(\Omega)$ of the equation*

$$div\,(A\nabla v) = 0$$

belongs to the Hölder space $C^{0,\alpha}_{loc}(\Omega)$ for some $0 < \alpha(n, \lambda, \Lambda) \leqslant 1$.

Solution of the Hilbert's 19th problem - (J. Moser)

- Sobolev embedding
- Caccioppoli inequality
- reverse Hölder inequality
- iteration of r.H.i.
- John–Nirenberg inequality
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Reverse Hölder inequality

$$\langle v^{2n/(n-2)} \rangle_{B_{r'}}^{(n-2)/n} \lesssim \langle v^2 \rangle_{B_r}, \qquad r' < r < 2r'.$$

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Complex case fails: Maz'ya–Nazarov–Plamenevskij (1982)

Existence of weak solutions to an elliptic equation which are not locally Hölder continuous, $n \geqslant 5$.

Dindoš-Pipher theorems (December 2016)

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Theorem 1 (Reverse Hölder inequality)

Suppose that $u \in H^1_{loc}(\Omega)$ is a weak solution to $\operatorname{div}(A\nabla u) = 0$ in Ω. Let

$$p_0 := \inf\{p > 1\,;\, A \text{ is } p - \textit{elliptic}\}.$$

Then, for any $B_{4r}(x) \subset \Omega$,

$$\langle |u|^p \rangle_{B_r(x)}^{1/p} \lesssim \langle |u|^q \rangle_{B_{2r}(x)}^{1/q}$$

for all $p, q \in (p_0, p_0' n/(n-2))$.
The implied constants depend on the p-ellipticity constants, n, Λ, but not on x, r, u.

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Mayboroda (2010): sharpness of the range of p.

Dindoš-Pipher theorems (December 2016)

Theorem 2 (Caccioppoli estimate)

Under the above assumptions we have, for $p \in (p_0, p_0')$,

$$\int_{B_r(x)} |\nabla u|^2 |u|^{p-2}\, dm \;\lesssim\; r^{-2} \int_{B_{2r}(x)} |u|^p\, dm.$$

Application:
solvability of the L^p Dirichlet boundary value problem for
$u \mapsto \operatorname{div}(A\nabla u)$
(again assuming p-ellipticity).

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Here $\alpha, \beta \in \mathbb{R}^n$ and $1/p + 1/q = 1$.

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Key assumption:

$$\Delta_p(A) > 0$$

That is, $\exists\ C > 0$ such that p.p. $x \in \Omega$ we have

$$\Re\langle A(x)\xi, \mathcal{J}_p\xi\rangle \geqslant C|\xi|^2 , \qquad \forall \xi \in \mathbb{C}^n.$$

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Obvious: $\Delta_2(A) > 0 \iff$ (uniform strict) ellipticity.

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If A is real then $\Delta_p(A) > 0$ for all $p > 1$.

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For any $A \in \mathcal{A}_n$ set

$$\mu(A) := \operatorname{ess\,inf} \Re \frac{\langle A(x)\xi, \xi \rangle}{|\langle A(x)\xi, \bar{\xi} \rangle|}\,;$$

ess inf over all $x \in \Omega$ and all $\xi \in \mathbb{C}^n$ for which $\langle A(x)\xi, \bar{\xi} \rangle \neq 0$.
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Immediate: $\lambda/\Lambda \leqslant \mu(A) \leqslant 1$.

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Study of power functions was motivated by our attempts to understand convexity of a particular **Bellman function** due to Nazarov and Treil, which comprises tensor products of power functions.

This was in turn pursued as a part of our (D.–Volberg 2011, Carbonaro–D. 2015) efforts to prove **bilinear embedding theorem** for arbitrary complex accretive matrices A.

Bilinear embedding

A typical example:

$$\int_0^\infty \int_\Omega \left|\nabla_x e^{-tL} f(x)\right| \left|\nabla_x e^{-tL} g(x)\right| \, d\mu(x) \, dt \lesssim \|f\|_p \|g\|_q$$

Proof: study of the monotonicity of the **heat flow**

$$t \mapsto \int_\Omega Q\left(e^{-tL} f, e^{-tL} g\right) d\mu$$

where $Q : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ should admit adequate:

- size estimate
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The best (known) example for our purpose: the Nazarov–Treil function.

Bellman function method: Nazarov–Treil–Volberg 1994

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An early concrete example:

Nazarov–Treil (1995)

Fix $p \geqslant 2$ and $\delta > 0$. Write $q = p/(p-1)$. Introduce $\wp = \wp_{p,\delta} : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by

$$\wp(u,v) = u^p + v^q + \delta \begin{cases} u^2 v^{2-q} & ; \quad u^p \leqslant v^q \\ \dfrac{2}{p}\, u^p + \left(\dfrac{2}{q} - 1\right) v^q & ; \quad u^p \geqslant v^q \,. \end{cases}$$

The *Bellman function*: $Q = Q_{p,\delta} : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{R}_+$,

$$Q(\zeta,\eta) := \wp(|\zeta|,|\eta|)\,.$$

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What is convexity?

Generalized Hessians

Suppose $F : \mathbb{C}\backslash\{0\} \to \mathbb{R}$ is differentiable and $A \in \mathbb{C}^{n,n}$. Introduce the identification operator $\mathcal{V} : \mathbb{C} \to \mathbb{R}^2$ by

$$\mathcal{V}(u + iv) = (u, v).$$

For $\zeta \in \mathbb{C}\backslash\{0\}$ and $\xi \in \mathbb{C}^n$ define (in **block notation**) the *generalized Hessian of F associated with A* by

$$H_F^A[\zeta;\xi] = \left\langle \mathrm{Hess}(F \circ \mathcal{V}^{-1}; \mathcal{V}(\zeta)) \begin{bmatrix} \Re\xi \\ \Im\xi \end{bmatrix}, \begin{bmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{bmatrix} \begin{bmatrix} \Re\xi \\ \Im\xi \end{bmatrix} \right\rangle_{\mathbb{R}^{2n}}$$

Basically, $H_F^A[\zeta;\xi]$ is the quadratic form corresponding to the matrix $A^T[d^2F(\zeta) \otimes I_n]$ and applied to ξ.

We say that A is convex with respect to F if $H_F^A[\zeta;\xi] > 0$ uniformly.

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Origin of p-ellipticity

$$\Delta_p(A) = \Delta_q(A) = \frac{2}{p^2} \operatorname*{ess\,inf}_{x \in \Omega} \min_{|\xi|=1} \min_{|\zeta|=1} H_{F_p}^{A(x)}[\zeta;\xi]$$

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$$= \frac{2}{p^2} \operatorname*{ess\,inf}_{x \in \Omega} \min_{|\xi|=1} H_{F_p}^{A(x)}[1;\xi]\,.$$

We find:

$$H_{F_p}^{A(x)}[1;\xi] = p^2\, \Re\langle A(x)\xi, \mathcal{J}_q \xi\rangle_{\mathbb{C}^n}\,.$$

Convexity of power functions – earlier cases (review)

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\begin{array}{ll}
A = I & (Nazarov - Treil\ 1995) \\
A\ real & (D. - Volberg\ 2011) \\
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In particular,

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This was essential for solving the problem of the optimal **holomorphic functional calculus** on L^p in sectors for arbitrary generators of symmetric contraction semigroups (Carbonaro–D. 2013) and nonsymmetric OU operators (Carbonaro–D. 2016).

Square functions

Bilinear integrals are dominated by vertical square functions:

$$\int_0^\infty \int_{\mathbb{R}^n} |\nabla_x e^{-tL_A} f(x)|\, |\nabla_x e^{-tL_B} g(x)|\, dx\, dt \leqslant \|G_{L_A} f\|_p \|G_{L_B} g\|_q\,,$$

where

$$G_L u(x) := \Big(\int_0^\infty \left|\nabla e^{-tL} u(x)\right|^2 dt\Big)^{1/2}.$$

Bilinear integrals are also dominated by conical square functions:

$$\int_0^\infty \int_{\mathbb{R}^n} |\nabla_x e^{-tL_A} f(x)|\, |\nabla_x e^{-tL_B} g(x)|\, dx\, dt \lesssim_n \|\mathfrak{g}_{L_A}(f)\|_p \|\mathfrak{g}_{L_B}(g)\|_q\,,$$

where, with $V_x = \{(y,t) \in \mathbb{R}^n \times (0,\infty)\,;\, |x-y| < \sqrt{t}\}$,

$$\mathfrak{g}_L(u)(x) = \left(\iint_{V_x} \left|\nabla_y (e^{-tL} u)(y)\right|^2 \frac{dy\, dt}{t^{n/2}}\right)^{1/2}.$$

(Fefferman-Stein 1972, Coifman-Meyer-Stein 1985.)

Bilinear embedding for complex accretive matrices

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Theorem (Carbonaro–D. 2015)

Suppose $p > 1$, $A, B \in \mathcal{A}_{\lambda,\Lambda}(\mathbb{R}^n)$ satisfy $\Delta_p := \Delta_p(A, B) > 0$. Then and all $f, g \in C_c^\infty(\mathbb{R}^n)$ we have

$$\int_0^\infty \int_{\mathbb{R}^n} |\nabla_x e^{-tL_A} f(x)|\, |\nabla_x e^{-tL_B} g(x)|\, dx\, dt \leqslant \frac{20}{\Delta_p} \cdot \frac{\Lambda}{\lambda} \|f\|_p \|g\|_q \,.$$

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$$\int_0^\infty \int_{\mathbb{R}^n} |\nabla_x e^{-tL_A} f(x)| \, |\nabla_x e^{-tL_B} g(x)| \, dx \, dt \leqslant \frac{20}{\Delta_p} \cdot \frac{\Lambda}{\lambda} \|f\|_p \|g\|_q \, .$$

Connection with contractivity of e^{-tL} on L^p?

Cialdea–Maz'ya theorems (2005)

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Let $A \in \mathcal{A}(\Omega)$ is such that

- $A \in C^1(\overline{\Omega})$ for some bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently regular boundary.

Take $p > 1$. Assume also that for all $x \in \Omega$,

$$\frac{4}{pq}\langle \Re A(x)\alpha, \alpha\rangle + \langle \Re A(x)\beta, \beta\rangle \\ + 2\left\langle \left(\frac{1}{p}\Im A(x) + \frac{1}{q}\Im A^*(x)\right)\alpha, \beta\right\rangle \geqslant 0 \qquad \forall \alpha, \beta \in \mathbb{R}^n;$$

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Question (Cialdea 2010):
generalize these results beyond the restrictions posed by the above smoothness and symmetry conditions.

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The last remaining case is fundamentally different, because when the div-zero condition fails, the Cialdea–Maz'ya criterion is in general **not** equivalent to the contractivity of $\exp(-tL_A)$ on $L^p(\Omega)$, not even for $A \in C^\infty(\mathbb{R}^n)$.

Proposition (sufficiency condition in Cialdea–Maz'ya)

Take $A = U + iV \in \mathcal{A}(\Omega)$ and $p > 1$. TFAE:

- p.p. $x \in \Omega$:

$$\frac{4}{pq}\langle \Re A(x)\alpha, \alpha\rangle + \langle \Re A(x)\beta, \beta\rangle + 2\left\langle \left(\frac{1}{p}\Im A(x) + \frac{1}{q}\Im A^*(x)\right)\alpha, \beta\right\rangle \geqslant 0 \qquad \forall \alpha, \beta \in \mathbb{R}^n;$$

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Our findings (cont.)

Theorem (Carbonaro–D. 2016)

Suppose that $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ is open, $A \in \mathcal{A}(\Omega)$ and $p > 1$. Consider the following statements:

(a) $\Delta_p(A) \geqslant 0$;

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- *if $div(\Im A)_a^{(k)} \neq 0$ for some $k \in \{1, \ldots, n\}$, then, in general, (c) $\not\Rightarrow$ (b).*

Ingredients of our proofs (cf. L^p-contractivity of e^{-tL_A})

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Let $\mathcal{D}_p(\mathfrak{a}) = \{u \in H_0^1(\Omega)\,;\, |u|^{p-2}u \in H_0^1(\Omega)\}$.

Nittka's theorem (2012)

$\|e^{-tL_A}\|_{p\to p} \leqslant 1$ if and only if

$$\int_\Omega \Re\langle A\nabla f, \nabla(|f|^{p-2}f)\rangle_{\mathbb{C}^n} \geqslant 0 \qquad \forall f \in \mathcal{D}_p(\mathfrak{a})\,.$$

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uniform positivity of $H^B_{F_p} \Leftrightarrow \Delta_p(B) \geqslant 0$

Consequences of p-ellipticity (summary)

p-ellipticity lies at the junction of:

1. convexity of power functions
   (uniform positivity of the Hessian forms $A^T[d^2|\zeta|^p \otimes I_n]$)
2. dimension-free bilinear embedding
3. L^p-contractivity of semigroups associated with elliptic div-form operators with (nonsmooth) complex coefficients
4. holomorphic functional calculus for generators of symmetric contraction semigroups on σ-finite spaces (p-ellipticity of $e^{i\phi}I$) and nonsymetric OU (p-ellipticity of $e^{i\phi}B$, B real)
5. (Dindoš-Pipher 2016)
   regularity theory of elliptic PDE with complex coefficients
   (reverse Hölder inequalities for solutions of $L_A u$ for **complex** matrices A).

Main literature

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