Toeplitz operators on the symmetrized bidisc

(A joint work with T. Bhattacharyya and B. K. Das)

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A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. 213 (1963) 89-102.

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$\Gamma := \overline{\mathbb{G}} = \{(z_1 + z_2, z_1 z_2) : |z_1| \le 1, |z_2| \le 1\}.$

People who have worked on this domain includes

J. Agler,	N. Young,	P. Pflug,	W. Zwonek,
L. Kosinski,	C. Costara,	Z. Lykova,	G. Bharali,
O. Shalit,	T. Bhattacharyya,	J. Sarkar,	S. Pal,
S. Biswas,	S. ShyamRoy,	S. Lata.	and B. K. Das.

The Hardy space $H^2(\mathbb{G})$ of the symmetrized bidisc is the vector space of those holomorphic functions f on $\mathbb{G}$ which satisfy

$$\textsf{sup}_{0<r<1}\int_{\mathbb{T}\times\mathbb{T}} |f\circ\pi(r\,e^{i\theta_1}, r\,e^{i\theta_2})|^2 |J(r\,e^{i\theta_1}, r\,e^{i\theta_2})|^2 d\theta_1 d\theta_2 < \infty$$

where J is the complex Jacobian of the symmetrization map π and $d\theta_i$ is the normalized Lebesgue measure on the unit circle $\mathbb{T} = \{\alpha : |\alpha| = 1\}$ for all $i = 1, 2$.

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where J is the complex Jacobian of the symmetrization map π and $d\theta_i$ is the normalized Lebesgue measure on the unit circle $\mathbb{T} = \{\alpha : |\alpha| = 1\}$ for all $i = 1, 2.$ The norm of $f \in H^2(\mathbb{G})$ is defined to be

$$\frac{1}{\|J\|}\Big\{\text{sup}_{0<r<1} \int_{\mathbb{T}\times\mathbb{T}} |f\circ\pi(r\,e^{i\theta_1}, r\,e^{i\theta_2})|^2 |J(r\,e^{i\theta_1}, r\,e^{i\theta_2})|^2 d\theta_1 d\theta_2\Big\}^{1/2},$$

where $\|J\|^2 = \int_{\mathbb{T}\times\mathbb{T}} |J(\,e^{i\theta_1},\, e^{i\theta_2})|^2 d\theta_1 d\theta_2.$

The distinguished boundary of Γ is

$$b\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| = 1 = |z_2|\} = \pi(\mathbb{T} \times \mathbb{T}).$$

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The Hilbert space $L^2(b\Gamma)$ consists of the following functions:

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Theorem

The space $H^2(\mathbb{G})$ sits isometrically inside the space $L^2(b\Gamma)$.

Let $L^\infty(b\Gamma)$ be the vectors space consisting of

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For a function φ in $L^\infty(b\Gamma)$, let M_φ be the operator on $L^2(b\Gamma)$ defined by

$$M_\varphi f(s,p) = \varphi(s,p) f(s,p), \text{ for all } f \text{ in } L^2(b\Gamma).$$

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Definition

For a function φ in $L^\infty(b\Gamma)$, the *Toeplitz operator* with symbol φ, denoted by T_φ, is defined by

$$T_\varphi f = PrM_\varphi f, \text{ for all } f \text{ in } H^2(\mathbb{G}).$$

Brown-Halmos relations on disc, polydisc and ball

A bounded operator T on $\underline{H^2(\mathbb{D})}$ is a Toeplitz operator if and only if

$$T_z^* T T_z = T.$$

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A bounded operator T on $\underline{H^2(\mathbb{B}_n)}$ is a Toeplitz operator if and only if

$$T_{z_1}^* T T_{z_1} + T_{z_2}^* T T_{z_2} + \cdots + T_{z_n}^* T T_{z_n} = T.$$

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$$T_s^* T T_p = T T_s \text{ and } T_p^* T T_p = T.$$

Corollary

If T commutes with both T_s and T_p, then T is a Toeplitz operator.

Definition

A Toeplitz operator with symbol φ is called an *analytic Toeplitz operator* if φ is in $H^\infty(\mathbb{G})$.

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A couple of definitions

Definition (Agler-Young, 2003)

A commuting pair (R, U) of bounded normal operators on a Hilbert space is called Γ-**unitary** if $\sigma(R, U)$ is contained in $b\Gamma$.

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Example

The pair (T_s, T_p) on $H^2(\mathbb{G})$ is a Γ-isometry with (M_s, M_p) on $L^2(b\Gamma)$ as its minimal Γ-unitary extension.

Generalized Toeplitz operators

Recall that a Toeplitz operator on $H^2(\mathbb{G})$ satisfies the Brown-Halmos relations with respect to the Γ-isometry (T_s, T_p), i.e., $T_s^* T T_p = T T_s$ and $T_p^* T T_p = T,$ and vice versa.

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Definition

Given a Γ-isometry (S, P) on a Hilbert space $\mathcal{H}$, we say that a bounded operator T on $\mathcal{H}$ is an (S, P)–Toeplitz operator, if it satisfies the Brown-Halmos relations with respect to the Γ-isometry (S, P) i.e.,

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Observations:

(1) Both S and P are (S, P)–Toeplitz operators. This is because every Γ-isometry (S, P) satisfies $S^*P = S$ and $P^*P = I$.

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Definition

Given a Γ-isometry (S, P) on a Hilbert space $\mathcal{H}$, we say that a bounded operator T on $\mathcal{H}$ is an <u>(S, P)–Toeplitz operator</u>, if it satisfies the Brown-Halmos relations with respect to the Γ-isometry (S, P) i.e.,

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<u>Observations:</u>

(1) Both S and P are (S, P)–Toeplitz operators. This is because every Γ-isometry (S, P) satisfies $S^*P = S$ and $P^*P = I$.

(2) Any operator commuting with (S, P) is an (S, P)–Toeplitz operator.

B. Prunaru, *Some exact sequences for Toeplitz algebras of spherical isometries*, Proc. Amer. Math. Soc. 135 (2007), 3621-3630.

Characterization

Theorem

Let (S, P) on $\mathcal{H}$ be a Γ-isometry and (R, U) on $\mathcal{K}$ be its minimal Γ-unitary extension.

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there exists an operator Y in the commutant of the von-Neumann algebra generated by $\{R, U\}$ such that $X = P_{\mathcal{H}} Y|_{\mathcal{H}}$ and $\|Y\| = \|X\|$.

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Observation: Choose (S,P) to be (T_s,T_p). Then (R,U) is (M_s,M_p) and hence $Y = M_\varphi$, for some $\varphi \in L^\infty(b\Gamma)$. Note that

$$M_\varphi = \begin{array}{c} \\ H^2(\mathbb{G}) \\ H^2(\mathbb{G})^\perp \end{array} \begin{array}{c} \begin{array}{cc} H^2(\mathbb{G}) & H^2(\mathbb{G})^\perp \end{array} \\ \left(\begin{array}{cc} T_\varphi & H^*_{\bar{\varphi}} \\ H_\varphi & * \end{array} \right) \end{array}$$

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Any operator X acting on $\mathcal{H}$ commuting with (S, P)

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X has a unique norm preserving extension Y acting on $\mathcal{K}$ commuting with (R, U).

Dual Toeplitz operators

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The dual Toeplitz operator for a $\varphi \in L^\infty(b\Gamma)$ is defined to be the following operator on $H^2(\mathbb{G})^\perp$,

$$DT_\varphi = (I - Pr)M_\varphi|_{H^2(\mathbb{G})^\perp}.$$

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With respect to the decomposition above,

$$M_\varphi = \begin{array}{c} \\ H^2(\mathbb{G}) \\ H^2(\mathbb{G})^\perp \end{array} \begin{array}{c} \begin{array}{cc} H^2(\mathbb{G}) & H^2(\mathbb{G})^\perp \end{array} \\ \begin{pmatrix} T_\varphi & H^*_{\overline{\varphi}} \\ H_\varphi & DT_\varphi \end{pmatrix} \end{array}$$

- Guediri, H., Dual Toeplitz operators on the sphere. Acta Math. Sin. (English Series) 29(9), 1791-1808 (2013)
- M. Didas and J. Eschmeier, *Dual Toeplitz operators on the spehere via sperical isometries*, Integr. Equat. Oper. Th. 83 (2015), 291-300.

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Lemma

The special pair $(DT_{\bar{s}}, DT_{\bar{p}})$ *is a* Γ*-isometry with* $(M_{\bar{s}}, M_{\bar{p}})$ *as its minimal* Γ*-unitary extension.*

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A bounded operator T on $H^2(\mathbb{G})^{\perp}$ is a dual Toeplitz operator

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A bounded operator T *on* $H^2(\mathbb{G})^{\perp}$ *is a dual Toeplitz operator*

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it is a $(DT_{\bar{s}}, DT_{\bar{p}})$*–Toeplitz operator, i.e., it satisfies the Brown-Halmos relations with respect to* $(DT_{\bar{s}}, DT_{\bar{p}})$*.*

Thank you for your attention.