Frames and operator representations of frames

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HATA: Harmonic Analysis - Theory and Applications
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Frames and overview of the talk

- If a sequence \( \{f_k\}_{k=1}^{\infty} \) in a Hilbert spaces \( \mathcal{H} \) is a frame, there exists another frame \( \{g_k\}_{k=1}^{\infty} \) such that

\[
f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \quad \forall f \in \mathcal{H}.
\]

Similar to the decomposition in terms of an orthonormal basis, but MUCH MORE flexible.
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Similar to the decomposition in terms of an orthonormal basis, but MUCH MORE flexible.

- We will consider representations of frames on the form

\[
    \{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty} = \{f_1, Tf_1, T^2 f_1, \ldots \},
\]

where \( T : \mathcal{H} \rightarrow \mathcal{H} \) is a linear operator, possibly bounded.
Frames and overview of the talk

- If a sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert spaces $\mathcal{H}$ is a frame, there exists another frame $\{g_k\}_{k=1}^{\infty}$ such that

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \forall f \in \mathcal{H}.$$ 

Similar to the decomposition in terms of an orthonormal basis, but MUCH MORE flexible.

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$$\{f_k\}_{k=1}^{\infty} = \{T^nf_1\}_{n=0}^{\infty} = \{f_1, Tf_1, T^2f_1, \cdots \},$$

where $T : \mathcal{H} \rightarrow \mathcal{H}$ is a linear operator, possibly bounded.

Main conclusion: Frame theory is operator theory, with several interesting and challenging open problems!
**Bessel sequences**

*Definition* A sequence \( \{ f_k \}_{k=1}^{\infty} \) in \( \mathcal{H} \) is called a Bessel sequence if there exists a constant \( B > 0 \) such that

\[
\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \| f \|^2, \quad \forall f \in \mathcal{H}.
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**Bessel sequences**

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\]

**Theorem** Let \( \{f_k\}_{k=1}^{\infty} \) be a sequence in \( \mathcal{H} \), and \( B > 0 \) be given. Then \( \{f_k\}_{k=1}^{\infty} \) is a Bessel sequence with Bessel bound \( B \) if and only if

\[
T : \{c_k\}_{k=1}^{\infty} \rightarrow \sum_{k=1}^{\infty} c_k f_k
\]

defines a bounded operator from \( \ell^2(\mathbb{N}) \) into \( \mathcal{H} \) and \( \|T\| \leq \sqrt{B} \).
Bessel sequences

Pre-frame operator or synthesis operator associated to a Bessel sequence:

\[ T : \ell^2(\mathbb{N}) \to \mathcal{H}, \ T\{c_k\}_{k=1}^\infty = \sum_{k=1}^{\infty} c_k f_k \]

The adjoint operator - the analysis operator:

\[ T^* : \mathcal{H} \to \ell^2(\mathbb{N}), \ T^* f = \{\langle f, f_k \rangle\}_{k=1}^\infty. \]

The frame operator:

\[ S : \mathcal{H} \to \mathcal{H}, \ S f = T T^* f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k. \]

The series defining \( S \) converges unconditionally for all \( f \in \mathcal{H}. \)
Frames

Definition: A sequence \( \{f_k\}_{k=1}^{\infty} \) in \( \mathcal{H} \) is a frame if there exist constants \( A, B > 0 \) such that

\[
A \|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}.
\]

A and \( B \) are called frame bounds.

Note:

- Any orthonormal basis is a frame;
- Example of a frame which is not a basis:

\[
\{e_1, e_1, e_2, e_3, \ldots \},
\]

where \( \{e_k\}_{k=1}^{\infty} \) is an ONB.
The frame decomposition

If \( \{f_k\}_{k=1}^{\infty} \) is a frame, the frame operator

\[
S : \mathcal{H} \rightarrow \mathcal{H}, \quad Sf = \sum \langle f, f_k \rangle f_k
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is well-defined, bounded, invertible, and self-adjoint.
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is well-defined, bounded, invertible, and self-adjoint.

**Theorem - the frame decomposition** Let \( \{f_k\}_{k=1}^{\infty} \) be a frame with frame operator \( S \). Then

\[
f = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k \rangle f_k, \quad \forall f \in \mathcal{H}.
\]

It might be difficult to compute \( S^{-1} \)!
The frame decomposition

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Theorem - the frame decomposition
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\]

It might be difficult to compute \( S^{-1} \)!

Important special case: If the frame \( \{f_k\}_{k=1}^{\infty} \) is tight, \( A = B \), then \( S = AI \) and

\[
f = \frac{1}{A} \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \quad \forall f \in \mathcal{H}.
\]
General dual frames

A frame which is not a basis is said to be *overcomplete*.

**Theorem:** Assume that \( \{ f_k \}_{k=1}^\infty \) is an overcomplete frame. Then there exist frames
\[
\{ g_k \}_{k=1}^\infty \neq \{ S^{-1} f_k \}_{k=1}^\infty
\]
for which
\[
f = \sum_{k=1}^\infty \langle f, g_k \rangle f_k = \sum_{k=1}^\infty \langle f, S^{-1} f_k \rangle f_k, \ \forall f \in \mathcal{H}.
\]

- \( \{ g_k \}_{k=1}^\infty \) is called a *dual frame* of \( \{ f_k \}_{k=1}^\infty \).
- The *excess* of a frame is the maximal number of elements that can be removed such that the remaining set is still a frame. The excess equals \( \dim N(T) \) - the dimension of the kernel of the synthesis operator.
- When the excess is large, the set of dual frames is large.
General dual frames

Note: Let \( \{f_k\}_{k=1}^{\infty} \) be a Bessel sequence with pre-frame operator

\[
T : \mathcal{H} \to \ell^2(\mathbb{N}), \quad T \{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k \quad \quad [T^* f = \{\langle f, f_k \rangle\}_{k=1}^{\infty}]
\]

and \( \{g_k\}_{k=1}^{\infty} \) be a Bessel sequence with pre-frame operator

\[
U : \mathcal{H} \to \ell^2(\mathbb{N}), \quad U \{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k g_k \quad \quad [U^* f = \{\langle f, g_k \rangle\}_{k=1}^{\infty}]
\]

Then \( \{f_k\}_{k=1}^{\infty} \) and \( \{g_k\}_{k=1}^{\infty} \) are dual frames if and only if

\[
f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \quad \forall f \in \mathcal{H},
\]

i.e., if and only if

\[
TU^* = I.
\]
Key tracks in frame theory:

- Frames in finite-dimensional spaces;
- Frames in general separable Hilbert spaces
- Concrete frames in concrete Hilbert spaces:
  - Gabor frames in $L^2(\mathbb{R}), L^2(\mathbb{R}^d)$;
  - Wavelet frames;
  - Shift-invariant systems, generalized shift-invariant (GSI) systems;
  - Shearlets, etc.
- Frames in Banach spaces;
- (GSI) Frames on LCA groups
- Frames via integrable group representations, coorbit theory.
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An Introduction to frames and Riesz bases, 2.edition, Birkhäuser 2016
Towards concrete frames - operators on $L^2(\mathbb{R})$

*Translation by* $a \in \mathbb{R}$: $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(T_a f)(x) = f(x - a)$.

*Modulation by* $b \in \mathbb{R}$: $E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(E_b f)(x) = e^{2\pi ibx} f(x)$.

All these operators are unitary on $L^2(\mathbb{R})$. 
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All these operators are unitary on $L^2(\mathbb{R})$.

**Gabor systems in $L^2(\mathbb{R})$:** have the form

$$\{e^{2\pi imbx}g(x - na)\}_{m,n \in \mathbb{Z}}$$

for some $g \in L^2(\mathbb{R})$, $a, b > 0$. Short notation:

$$\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} = \{e^{2\pi imbx}g(x - na)\}_{m,n \in \mathbb{Z}}$$
Gabor systems in $L^2(\mathbb{R})$

It is known how to construct frames and dual pairs of frames with the Gabor structure
\[
\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx}g(x- na)\}_{m,n\in\mathbb{Z}}
\]

- Typical choices of $g$: B-splines or the Gaussian.
- $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ can only be a frame if $ab \leq 1$.
- If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame, then it is a basis if and only if $ab = 1$.
- Gabor frames $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ are always linearly independent, and they have infinite excess if $ab < 1$. 
Dynamical Sampling

Introduced in papers by Aldroubi, Davis & Krishtal, and Aldroubi, Cabrelli, Molter & Tang. Further developed in papers by

Aceska, Aldroubi, Cabrelli, Çakmak, Kim, Molter, Paternostro, Petrosyan, Philipp.

Let $\mathcal{H}$ denote a Hilbert space, and $A$ a class of operators $T : \mathcal{H} \rightarrow \mathcal{H}$. For $T \in A$ and $\varphi \in \mathcal{H}$, consider the iterated system

$$\{T^n \varphi\}_{n=0}^{\infty} = \{\varphi, T\varphi, T^2 \varphi \cdots \}.$$
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$$\{T^n \varphi\}_{n=0}^{\infty} = \{\varphi, T \varphi, T^2 \varphi, \ldots\}.$$  

Key questions:

- Can $\{T^n \varphi\}_{n=0}^{\infty}$ be a basis for $\mathcal{H}$ for some $T \in A$, $\varphi \in \mathcal{H}$?
- Can $\{T^n \varphi\}_{n=0}^{\infty}$ be a frame for $\mathcal{H}$ for some $T \in A$, $\varphi \in \mathcal{H}$?
Consider a bounded operator $T : \mathcal{H} \to \mathcal{H}$.

Recall:

- A vector $\varphi \in \mathcal{H}$ is cyclic if $\overline{\text{span}} \{T^n \varphi\}_{n=0}^{\infty} = \mathcal{H}$. 
Consider a bounded operator $T : \mathcal{H} \rightarrow \mathcal{H}$.

**Recall:**

- A vector $\varphi \in \mathcal{H}$ is cyclic if $\overline{\text{span}}\{T^n \varphi\}_{n=0}^\infty = \mathcal{H}$.

  This is much weaker than the condition that $\{T^n \varphi\}_{n=0}^\infty$ is a frame.
Consider a bounded operator $T : \mathcal{H} \to \mathcal{H}$.

Recall:

- A vector $\varphi \in \mathcal{H}$ is cyclic if $\overline{\text{span}}\{T^n \varphi\}_{n=0}^\infty = \mathcal{H}$. This is much weaker than the condition that $\{T^n \varphi\}_{n=0}^\infty$ is a frame.
- A vector $\varphi \in \mathcal{H}$ is hypercyclic if $\{T^n \varphi\}_{n=0}^\infty$ is dense in $\mathcal{H}$. 
Consider a bounded operator $T : \mathcal{H} \rightarrow \mathcal{H}$.

**Recall:**

- A vector $\varphi \in \mathcal{H}$ is cyclic if $\text{span}\{T^n \varphi\}_{n=0}^\infty = \mathcal{H}$. This is much weaker than the condition that $\{T^n \varphi\}_{n=0}^\infty$ is a frame.

- A vector $\varphi \in \mathcal{H}$ is hypercyclic if $\{T^n \varphi\}_{n=0}^\infty$ is dense in $\mathcal{H}$. This is too strong in the frame context - it excludes that $\{T^n \varphi\}_{n=0}^\infty$ is a Bessel sequence.
Dynamical Sampling

Recall the key questions in dynamical sampling:

- Can \( \{ T^n \varphi \}_{n=0}^{\infty} \) be a basis for \( \mathcal{H} \) for some \( T \in A, \varphi \in \mathcal{H} \)?
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Dual approach by C. & Marzieh Hasannasab:

- When does a given frame \( \{f_k\}_{k=1}^\infty \) has a representation

\[
\{f_k\}_{k=1}^\infty = \{T^n \varphi\}_{n=0}^\infty
\]

for some operator \( T : \mathcal{H} \to \mathcal{H} \)?
Recall the key questions in dynamical sampling:

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- When does a given frame \( \{ f_k \}_{k=1}^{\infty} \) has a representation
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  \]
  for some operator \( T : \mathcal{H} \to \mathcal{H} \)?
- Under what conditions is such a representation possible with a bounded operator \( T \)?
- What are the properties of such frames?
- What are the properties of the relevant operators \( T \)?
Dynamical sampling

Plan for the rest of the talk:

- A sample of results from the literature
- A characterization of the frames that have a representation
  \( \{f_k\}_{k=1}^{\infty} = \{T^n \varphi\}_{n=0}^{\infty} \) for some operator \( T : \text{span}\{f_k\}_{k=1}^{\infty} \to \mathcal{H} \).
- Characterizations of the case where \( T \) can be chosen to be bounded.
- Properties of \( \{f_k\}_{k=1}^{\infty} \) and properties of \( T \).
- Open problems.
Results from the literature

- If $T$ is normal, then $\{T^n \varphi\}_{n=0}^{\infty}$ is not a basis (Aldroubi, Cabrelli, Çakmak, Molter, Petrosyan).
- If $T$ is unitary, then $\{T^n \varphi\}_{n=0}^{\infty}$ is not a frame (Aldroubi, Petrosyan).
- If $T$ is compact, then $\{T^n \varphi\}_{n=0}^{\infty}$ is not a frame (C., Hasannasab, Rashidi).
A positive result from the literature

Example (Aldroubi, Cabrelli, Molter, Tang) Consider an operator $T$ of the form $T = \sum_{k=1}^{\infty} \lambda_k P_k$, where $P_k, k \in \mathbb{N}$, are rank 1 orthogonal projections such that $P_j P_k = 0$, $j \neq k$, $\sum_{k=1}^{\infty} P_k = I$, and $|\lambda_k| < 1$ for all $k \in \mathbb{N}$.

- There exists an ONB $\{e_k\}_{k=1}^{\infty}$ such that
  
  $$Tf = \sum_{k=1}^{\infty} \lambda_k \langle f, e_k \rangle e_k, \ f \in \mathcal{H}.$$  

- Assume that $\{\lambda_k\}_{k=1}^{\infty}$ satisfies the Carleson condition, i.e.,
  
  $$\inf_k \prod_{j \neq k} \frac{|\lambda_j - \lambda_k|}{|1 - \lambda_j \lambda_k|} > 0.$$  

- Letting $\varphi := \sum_{k=1}^{\infty} \sqrt{1 - |\lambda_k|^2} e_k$, the family $\{T^n \varphi\}_{n=0}^{\infty}$ is a frame for $\mathcal{H}$.
- Concrete case: $\lambda_k = 1 - \alpha^{-k}$ for some $\alpha > 1$.  

The key question, formulated in $\ell^2(\mathbb{N})$

Motivated by all the talks on Hilbert/Hankel/Helson/Toeplitz matrices:

**Key question:** Identify a class of infinite non-diagonalizable matrices $A = (a_{m,n})_{m,n \geq 1}$ and some vector $\varphi \in \ell^2(\mathbb{N})$ such that the collection of vectors

$$\{A^n \varphi\}_{n=0}^\infty = \{\varphi, A\varphi, A^2 \varphi, \cdots \}$$

form a frame for $\ell^2(\mathbb{N})$. 
Existence of the representation $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$

Proposition (Hasannasab & C., 2016): Consider a frame $\{f_k\}_{k=1}^\infty$ for an infinite-dimensional Hilbert space $\mathcal{H}$. Then the following are equivalent:

- There exists a linear operator $T : \text{span}\{f_k\}_{k=1}^\infty \rightarrow \mathcal{H}$ such that

  \[
  \{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty,
  \]

- The sequence $\{f_k\}_{k=1}^\infty$ is linearly independent.
Boundedness of $T$ in the representation $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$

- If $\{f_k\}_{k=1}^\infty$ is a Bessel sequence, the synthesis operator

$$U : \ell^2(\mathbb{N}) \to \mathcal{H}, \quad U\{c_k\}_{k=1}^\infty := \sum_{k=1}^\infty c_k f_k$$

is well-defined and bounded.

- The kernel of $U$ is

$$\mathcal{N}(U) = \left\{ \{c_k\}_{k=1}^\infty \in \ell^2(\mathbb{N}) \mid \sum_{k=1}^\infty c_k f_k = 0 \right\}.$$

- Consider the right-shift operator $\mathcal{T}$ on $\ell^2(\mathbb{N})$, defined by

$$\mathcal{T}(c_1, c_2, \cdots) = (0, c_1, c_2, \cdots).$$
Boundedness of $T$ in the representation $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$

Theorem (Hasannasab & C., 2017): Consider a frame $\{f_k\}_{k=1}^\infty$. Then the following are equivalent:

(i) The frame has a representation $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$ for some bounded operator $T : \mathcal{H} \to \mathcal{H}$. 
Boundedness of $T$ in the representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$

**Theorem (Hasannasab & C., 2017):** Consider a frame $\{f_k\}_{k=1}^{\infty}$. Then the following are equivalent:

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(ii) $\{f_k\}_{k=1}^{\infty}$ is linearly independent and the kernel $\mathcal{N}(U)$ of the synthesis operator $U$ is invariant under the right-shift operator $T$. 
Boundedness of $T$ in the representation $\{f_k\}_{k=1}^{\infty} = \{T^nf_1\}_{n=0}^{\infty}$

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(iii) For some dual frame $\{g_k\}_{k=1}^{\infty}$ (and hence all),

$$f_{j+1} = \sum_{k=1}^{\infty} \langle f_j, g_k \rangle f_{k+1}, \quad \forall j \in \mathbb{N}.$$
Theorem (Hasannasab & C., 2017): Consider a frame \( \{f_k\}_{k=1}^{\infty} \). Then the following are equivalent:

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\[
    f_{j+1} = \sum_{k=1}^{\infty} \langle f_j, g_k \rangle f_{k+1}, \quad \forall j \in \mathbb{N}.
\]

In the affirmative case, letting \( \{g_k\}_{k=1}^{\infty} \) denote an arbitrary dual frame of \( \{f_k\}_{k=1}^{\infty} \), the operator \( T \) has the form

\[
    Tf = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_{k+1}, \quad \forall f \in \mathcal{H}.
\]
Boundedness of $T$ in the representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$

Corollary: Any Riesz basis $\{f_k\}_{k=1}^{\infty}$ has the form $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ for some bounded operator $T : \mathcal{H} \to \mathcal{H}$.

Surprisingly, the availability of such a representation fails for frames with finite excess:

Proposition: Assume that $\{f_k\}_{k=1}^{\infty}$ is a frame and $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ for a linear operator $T$. If $\{f_k\}_{k=1}^{\infty}$ has finite and strictly positive excess, then $T$ is unbounded.
Boundedness of $T$ in the representation $\{f_k\}_{k=1}^{\infty} = \{T^nf_1\}_{n=0}^{\infty}$

Note: The properties of a frame with a representation $\{f_k\}_{k=1}^{\infty} = \{T^nf_1\}_{n=0}^{\infty}$ (linear independence and infinite excess) match precisely the properties of Gabor frames $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ for $ab < 1$!
Boundedness of $T$ in the representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$

Note: The properties of a frame with a representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ (linear independence and infinite excess) match precisely the properties of Gabor frames $\{E_{mb} T_{nag}\}_{m,n \in \mathbb{Z}}$ for $ab < 1$!

Can a Gabor frame $\{E_{mb} T_{nag}\}_{m,n \in \mathbb{Z}}$ with $ab < 1$ be represented on this form with a bounded operator $T$?
Boundedness of $T$ in the representation $\{f_k\}_{k=1}^\infty = \{T^nf_1\}_{n=0}^\infty$

Note: The properties of a frame with a representation $\{f_k\}_{k=1}^\infty = \{T^nf_1\}_{n=0}^\infty$ (linear independence and infinite excess) match precisely the properties of Gabor frames $\{E_{mb}T_{nag}\}_{m,n\in\mathbb{Z}}$ for $ab < 1$!

Can a Gabor frame $\{E_{mb}T_{nag}\}_{m,n\in\mathbb{Z}}$ with $ab < 1$ be represented on this form with a bounded operator $T$?

No!
Boundedness of $T$ in the representation $\{f_k\}_{k=1}^\infty = \{T^nf_1\}_{n=0}^\infty$

Note: The properties of a frame with a representation $\{f_k\}_{k=1}^\infty = \{T^nf_1\}_{n=0}^\infty$ (linear independence and infinite excess) match precisely the properties of Gabor frames $\{E_{mb}T_{nag}\}_{m,n\in\mathbb{Z}}$ for $ab < 1$!

Can a Gabor frame $\{E_{mb}T_{nag}\}_{m,n\in\mathbb{Z}}$ with $ab < 1$ be represented on this form with a bounded operator $T$?

No!

Question: Can a Gabor frame $\{E_{mb}T_{nag}\}_{m,n\in\mathbb{Z}}$ with $ab < 1$ be represented on the

$$\{E_{mb}T_{nag}\}_{m,n\in\mathbb{Z}} = \{a_nT^ng\}_{n=0}^\infty$$

for some scalars $a_n > 0$ and a bounded operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$?
A comment on the indexing

Let us index the frames by $\mathbb{Z}$ instead of $\mathbb{N}_0$:

- Frames with a representation $\{f_k\}_{k=1}^\infty = \{T^k f_0\}_{k \in \mathbb{Z}}$ have a similar characterization as for the index set $\mathbb{N}_0$.
- The conditions for boundedness of $T$ are similar.
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- The standard operators in applied harmonic analysis easily leads to such frames represented by bounded operators:
  - Translation: $\{T_k \varphi\}_{k \in \mathbb{Z}} = \{(T_1)^k \varphi\}_{k \in \mathbb{Z}}$;
  - Modulation: $\{E_{mb} \varphi\}_{m \in \mathbb{Z}} = \{(E_b)^m \varphi\}_{m \in \mathbb{Z}}$;
  - Scaling: $\{D_a \varphi\}_{j \in \mathbb{Z}} = \{(D_a)^j \varphi\}_{j \in \mathbb{Z}}$;
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**Question:** Consider a Gabor frame $\{E_{mb} T_{nag}\}_{m,n \in \mathbb{Z}}$ for $L^2(\mathbb{R})$ with $ab < 1$. Does the frame $\{E_{mb} T_{nag}\}_{m,n \in \mathbb{Z}}$ have a representation

$$\{E_{mb} T_{nag}\}_{m,n \in \mathbb{Z}} = \{T^n \varphi\}_{n=-\infty}^\infty$$

for a bounded operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and some $\varphi \in L^2(\mathbb{R})$?
Frames of the form \( \{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty} \) are special!

**Theorem (C., Hasannasab, Rashidi, 2017)** Assume that \( \{T^n \varphi\}_{n=0}^{\infty} \) is an overcomplete frame for some \( \varphi \in \mathcal{H} \) and some bounded operator \( T : \mathcal{H} \rightarrow \mathcal{H} \). Then the following hold:

(i) The image chain for the operator \( T \) has finite length \( q(T) \).

(ii) If \( N \in \mathbb{N} \), then \( T^N \varphi \in \overline{\text{span}}\{T^n \varphi\}_{n=N+1}^{\infty} \Leftrightarrow N \geq q(T) \).

For any \( N \geq q(T) \), let \( V := \overline{\text{span}}\{T^n \varphi\}_{n=N}^{\infty} \). Then the following hold:

(iii) The space \( V \) is independent of \( N \) and has finite codimension.

(iv) The sequence \( \{T^n \varphi\}_{n=N+\ell}^{\infty} \) is a frame for \( V \) for all \( \ell \in \mathbb{N}_0 \).

(v) \( V \) is invariant under \( T \), and \( T : V \rightarrow V \) is surjective.

(vi) If the null chain of \( T \) has finite length then \( T : V \rightarrow V \) is injective; in particular this is the case if \( T \) is normal.
A way to achieve boundedness - multiple operators

Theorem (Hasannasab & C., 2016) Consider a frame \( \{f_k\}_{k=1}^{\infty} \) which is norm-bounded below. Then there is a finite collection of vectors from \( \{f_k\}_{k=1}^{\infty} \), to be called \( \varphi_1, \ldots, \varphi_J \), and corresponding bounded operators \( T_j : \mathcal{H} \rightarrow \mathcal{H} \), such that

\[
\{f_k\}_{k=1}^{\infty} = \bigcup_{j=1}^{J} \{T_j^n \varphi_j\}_{n=0}^{\infty}.
\]

Remark: The assumption that \( \{f_k\}_{k=1}^{\infty} \) is norm-bounded below can not be removed.
References


