Frames and operator representations of frames

Ole Christensen
Joint work with Marzieh Hasannasab

HATA – DTU
DTU Compute, Technical University of Denmark

HATA: Harmonic Analysis - Theory and Applications
https://hata.compute.dtu.dk/
Ole Christensen
Jakob Lemvig
Mads Sielemann Jakobsen
Marzieh Hasannasab
Kamilla Haahr Nielsen
Ehsan Rashidi
Jordy van Velthoven

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Frames and overview of the talk

- If a sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert spaces $\mathcal{H}$ is a frame, there exists another frame $\{g_k\}_{k=1}^{\infty}$ such that

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \ \forall f \in \mathcal{H}.$$

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- We will consider representations of frames on the form

$$\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty} = \{f_1, Tf_1, T^2 f_1, \cdots\},$$

  where $T : \mathcal{H} \to \mathcal{H}$ is a linear operator, possibly bounded.

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Main conclusion: Frame theory is operator theory, with several interesting and challenging open problems!

Bessel sequences

Definition A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is called a Bessel sequence if there exists a constant $B > 0$ such that

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \, ||f||^2, \; \forall f \in \mathcal{H}.$$

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Theorem Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{H}$, and $B > 0$ be given. Then $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence with Bessel bound B if and only if

$$T : \{c_k\}_{k=1}^{\infty} \to \sum_{k=1}^{\infty} c_k f_k$$

defines a bounded operator from $\ell^2(\mathbb{N})$ into $\mathcal{H}$ and $||T|| \leq \sqrt{B}$.

Bessel sequences

Pre-frame operator or *synthesis operator* associated to a Bessel sequence:

$$T : \ell^2(\mathbb{N}) \to \mathcal{H},\ T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k$$

The adjoint operator - the *analysis operator:*

$$T^* : \mathcal{H} \to \ell^2(\mathbb{N}),\ \ T^* f = \{\langle f, f_k \rangle\}_{k=1}^{\infty}.$$

The *frame operator:*

$$S : \mathcal{H} \to \mathcal{H},\ \ Sf = TT^* f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k.$$

The series defining S converges unconditionally for all $f \in \mathcal{H}$.

Frames

Definition: A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is a *frame* if there exist constants $A, B > 0$ such that

$$A\,||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\,||f||^2,\ \forall f \in \mathcal{H}.$$

A and B are called *frame bounds.*

Note:

- Any orthonormal basis is a frame;
- Example of a frame which is not a basis:

$$\{e_1, e_1, e_2, e_3, \dots\},$$

where $\{e_k\}_{k=1}^{\infty}$ is an ONB.

The frame decomposition

If $\{f_k\}_{k=1}^{\infty}$ is a frame, the frame operator

$$S : \mathcal{H} \to \mathcal{H}, \; Sf = \sum \langle f, f_k \rangle f_k$$

is well-defined, bounded, invertible, and self-adjoint.

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Theorem - the frame decomposition Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S. Then

$$f = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k \rangle f_k, \quad \forall f \in \mathcal{H}.$$

It might be difficult to compute S^{-1} !

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Important special case: If the frame $\{f_k\}_{k=1}^{\infty}$ is tight, $A = B$, then $S = A\,I$ and

$$f = \frac{1}{A} \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \quad \forall f \in \mathcal{H}.$$

General dual frames

A frame which is not a basis is said to be *overcomplete*.

Theorem: *Assume that* $\{f_k\}_{k=1}^{\infty}$ *is an overcomplete frame. Then there exist frames*

$$\{g_k\}_{k=1}^{\infty} \neq \{S^{-1}f_k\}_{k=1}^{\infty}$$

for which

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k = \sum_{k=1}^{\infty} \langle f, S^{-1}f_k \rangle f_k, \ \forall f \in \mathcal{H}.$$

- $\{g_k\}_{k=1}^{\infty}$ is called a *dual frame* of $\{f_k\}_{k=1}^{\infty}$.
- The *excess* of a frame is the maximal number of elements that can be removed such that the remaining set is still a frame. The excess equals $\dim N(T)$ - the dimension of the kernel of the synthesis operator.
- When the excess is large, the set of dual frames is large.

General dual frames

Note: Let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence with pre-frame operator

$$T:\mathcal{H}\to\ell^2(\mathbb{N}),\ T\{c_k\}_{k=1}^{\infty}=\sum_{k=1}^{\infty}c_kf_k \qquad [T^*f=\{\langle f,f_k\rangle\}_{k=1}^{\infty}]$$

and $\{g_k\}_{k=1}^{\infty}$ be a Bessel sequence with pre-frame operator

$$U:\mathcal{H}\to\ell^2(\mathbb{N}),\ U\{c_k\}_{k=1}^{\infty}=\sum_{k=1}^{\infty}c_kg_k \qquad [U^*f=\{\langle f,g_k\rangle\}_{k=1}^{\infty}]$$

Then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are dual frames if and only if

$$f=\sum_{k=1}^{\infty}\langle f,g_k\rangle f_k,\ \forall f\in\mathcal{H},$$

i.e., if and only if

$$TU^*=I.$$

Key tracks in frame theory:

- Frames in finite-dimensional spaces;
- Frames in general separable Hilbert spaces
- Concrete frames in concrete Hilbert spaces:
  - Gabor frames in $L^2(\mathbb{R}), L^2(\mathbb{R}^d)$;
  - Wavelet frames;
  - Shift-invariant systems, generalized shift-invariant (GSI) systems;
  - Shearlets, etc.
- Frames in Banach spaces;
- (GSI) Frames on LCA groups
- Frames via integrable group representations, coorbit theory.

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An Introduction to frames and Riesz bases, 2.edition, Birkhäuser 2016

Towards concrete frames - operators on $L^2(\mathbb{R})$

Translation by $a \in \mathbb{R}$: $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (T_a f)(x) = f(x-a).$

Modulation by $b \in \mathbb{R}$: $E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (E_b f)(x) = e^{2\pi ibx} f(x).$

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Gabor systems in $L^2(\mathbb{R})$: have the form

$$\{e^{2\pi imbx} g(x-na)\}_{m,n\in\mathbb{Z}}$$

for some $g \in L^2(\mathbb{R}),\ a,b > 0$. Short notation:

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx} g(x-na)\}_{m,n\in\mathbb{Z}}$$

Gabor systems in $L^2(\mathbb{R})$

It is known how to construct frames and dual pairs of frames with the Gabor structure $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$

- Typical choices of g: B-splines or the Gaussian.
- $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ can only be a frame if $ab \leq 1$.
- If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame, then it is a basis if and only if $ab = 1$.
- Gabor frames $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ are always linearly independent, and they have infinite excess if $ab < 1$.

Dynamical Sampling

Introduced in papers by Aldroubi, Davis & Krishtal, and Aldroubi, Cabrelli, Molter & Tang. Further developed in papers by

Aceska, Aldroubi, Cabrelli, Çakmak, Kim,
Molter, Paternostro, Petrosyan, Philipp.

Let $\mathcal{H}$ denote a Hilbert space, and $\mathcal{A}$ a class of operators $T : \mathcal{H} \to \mathcal{H}$. For $T \in \mathcal{A}$ and $\varphi \in \mathcal{H}$, consider the iterated system

$$\{T^n\varphi\}_{n=0}^{\infty} = \{\varphi, T\varphi, T^2\varphi \cdots\}.$$

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Key questions:

- Can $\{T^n\varphi\}_{n=0}^{\infty}$ be a basis for $\mathcal{H}$ for some $T \in \mathcal{A}, \varphi \in \mathcal{H}$?
- Can $\{T^n\varphi\}_{n=0}^{\infty}$ be a frame for $\mathcal{H}$ for some $T \in \mathcal{A}, \varphi \in \mathcal{H}$?

Dynamical Sampling and standard operator theory

Consider a bounded operator $T : \mathcal{H} \to \mathcal{H}$.

Recall:

- A vector $\varphi \in \mathcal{H}$ is cyclic if $\overline{\text{span}}\{T^n \varphi\}_{n=0}^{\infty} = \mathcal{H}$.

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  This is much weaker than the condition that $\{T^n\varphi\}_{n=0}^\infty$ is a frame.
- A vector $\varphi \in \mathcal{H}$ is hypercyclic if $\{T^n\varphi\}_{n=0}^\infty$ is dense in $\mathcal{H}$.
  This is too strong in the frame context - it excludes that $\{T^n\varphi\}_{n=0}^\infty$ is a Bessel sequence.

Dynamical Sampling

Recall the key questions in dynamical sampling:

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Dual approach by C. & Marzieh Hasannasab:

- When does a given frame $\{f_k\}_{k=1}^{\infty}$ has a representation

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  for some operator $T : \mathcal{H} \to \mathcal{H}$?
- Under what conditions is such a representation possible with a bounded operator T?
- What are the properties of such frames?
- What are the properties of the relevant operators T?

Dynamical sampling

Plan for the rest of the talk:

- A sample of results from the literature
- A characterization of the frames that have a representation $\{f_k\}_{k=1}^{\infty} = \{T^n\varphi\}_{n=0}^{\infty}$ for some operator $T : \text{span}\{f_k\}_{k=1}^{\infty} \to \mathcal{H}$.
- Characterizations of the case where T can be chosen to be bounded.
- Properties of $\{f_k\}_{k=1}^{\infty}$ and properties of T.
- Open problems.

Results from the literature

- If T is normal, then $\{T^n\varphi\}_{n=0}^{\infty}$ is not a basis (Aldroubi, Cabrelli, Çakmak, Molter, Petrosyan).
- If T is unitary, then $\{T^n\varphi\}_{n=0}^{\infty}$ is not a frame (Aldroubi, Petrosyan).
- If T is compact, then $\{T^n\varphi\}_{n=0}^{\infty}$ is not a frame (C., Hasannasab, Rashidi).

A positive result from the literature

Example (Aldroubi, Cabrelli, Molter, Tang) Consider an operator T of the form $T = \sum_{k=1}^{\infty} \lambda_k P_k$, where $P_k, k \in \mathbb{N}$, are rank 1 orthogonal projections such that $P_j P_k = 0$, $j \neq k$, $\sum_{k=1}^{\infty} P_k = I$, and $|\lambda_k| < 1$ for all $k \in \mathbb{N}$.

- There exists an ONB $\{e_k\}_{k=1}^{\infty}$ such that

$$Tf = \sum_{k=1}^{\infty} \lambda_k \langle f, e_k \rangle e_k, \; f \in \mathcal{H}.$$

- Assume that $\{\lambda_k\}_{k=1}^{\infty}$ satisfies the Carleson condition, i.e.,

$$\inf_k \prod_{j \neq k} \frac{|\lambda_j - \lambda_k|}{|1 - \lambda_j \overline{\lambda_k}|} > 0.$$

- Letting $\varphi := \sum_{k=1}^{\infty} \sqrt{1 - |\lambda_k|^2} e_k$, the family $\{T^n \varphi\}_{n=0}^{\infty}$ is a frame for $\mathcal{H}$.
- Concrete case: $\lambda_k = 1 - \alpha^{-k}$ for some $\alpha > 1$.

The key question, formulated in $\ell^2(\mathbb{N})$

Motivated by all the talks on Hilbert/Hankel/Helson/Toeplitz matrices:

Key question: Identify a class of infinite non-diagonalizable matrices $A = (a_{m,n})_{m,n \geq 1}$ and some vector $\varphi \in \ell^2(\mathbb{N})$ such that the collection of vectors

$$\{A^n \varphi\}_{n=0}^{\infty} = \{\varphi, A\varphi, A^2\varphi, \cdots\}$$

form a frame for $\ell^2(\mathbb{N})$.

Existence of the representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$

Proposition (Hasannasab & C., 2016): Consider a frame $\{f_k\}_{k=1}^{\infty}$ for an infinite-dimensional Hilbert space $\mathcal{H}$. Then the following are equivalent:

- There exists a linear operator $T : \text{span}\{f_k\}_{k=1}^{\infty} \to \mathcal{H}$ such that

$$\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty},$$

- The sequence $\{f_k\}_{k=1}^{\infty}$ is linearly independent.

Boundedness of T in the representation $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$

- If $\{f_k\}_{k=1}^\infty$ is a Bessel sequence, the *synthesis operator*

$$U : \ell^2(\mathbb{N}) \to \mathcal{H},\ U\{c_k\}_{k=1}^\infty := \sum_{k=1}^\infty c_k f_k$$

  is well-defined and bounded.
- The kernel of U is

$$\mathcal{N}(U) = \left\{ \{c_k\}_{k=1}^\infty \in \ell^2(\mathbb{N}) \,\middle|\, \sum_{k=1}^\infty c_k f_k = 0 \right\}.$$

- Consider the right-shift operator $\mathcal{T}$ on $\ell^2(\mathbb{N})$, defined by

$$\mathcal{T}(c_1, c_2, \cdots) = (0, c_1, c_2, \cdots).$$

Boundedness of T in the representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$

Theorem (Hasannasab & C., 2017): Consider a frame $\{f_k\}_{k=1}^{\infty}$. Then the following are equivalent:

(i) The frame has a representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ for some bounded operator $T : \mathcal{H} \to \mathcal{H}$.

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(iii) For some dual frame $\{g_k\}_{k=1}^{\infty}$ (and hence all),

$$f_{j+1} = \sum_{k=1}^{\infty} \langle f_j, g_k \rangle f_{k+1}, \ \forall j \in \mathbb{N}.$$

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In the affirmative case, letting $\{g_k\}_{k=1}^{\infty}$ denote an arbitrary dual frame of $\{f_k\}_{k=1}^{\infty}$, the operator T has the form

$$Tf = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_{k+1}, \ \forall f \in \mathcal{H}.$$

Boundedness of T in the representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$

Corollary: Any Riesz basis $\{f_k\}_{k=1}^{\infty}$ has the form $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ for some bounded operator $T : \mathcal{H} \to \mathcal{H}$.

Surprisingly, the availability of such a representation fails for frames with finite excess:

Proposition: Assume that $\{f_k\}_{k=1}^{\infty}$ is a frame and $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ for a linear operator T. If $\{f_k\}_{k=1}^{\infty}$ has finite and strictly positive excess, then T is unbounded.

Boundedness of T in the representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$

Note: The properties of a frame with a representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ (linear independence and infinite excess) match precisely the properties of Gabor frames $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for $ab < 1$!

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Can a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with $ab < 1$ be represented on this form with a bounded operator T?

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No!

Question: Can a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with $ab < 1$ be represented on the

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} = \{a_n T^n g\}_{n=0}^\infty$$

for some scalars $a_n > 0$ and a bounded operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$?

A comment on the indexing

Let us index the frames by $\mathbb{Z}$ instead of $\mathbb{N}_0$:

- Frames with a representation $\{f_k\}_{k=1}^{\infty} = \{T^k f_0\}_{k\in\mathbb{Z}}$ have a similar characterization as for the index set $\mathbb{N}_0$.
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- The conditions for boundedness of T are similar.
- The standard operators in applied harmonic analysis easily leads to such frames represented by bounded operators:
    - Translation: $\{T_k\varphi\}_{k\in\mathbb{Z}} = \{(T_1)^k\varphi\}_{k\in\mathbb{Z}}$;
    - Modulation: $\{E_{mb}\varphi\}_{m\in\mathbb{Z}} = \{(E_b)^m\varphi\}_{m\in\mathbb{Z}}$;
    - Scaling: $\{D_{a^j}\varphi\}_{j\in\mathbb{Z}} = \{(D_a)^j\varphi\}_{j\in\mathbb{Z}}$;

A comment on the indexing

Let us index the frames by $\mathbb{Z}$ instead of $\mathbb{N}_0$:

- Frames with a representation $\{f_k\}_{k=1}^{\infty} = \{T^k f_0\}_{k\in\mathbb{Z}}$ have a similar characterization as for the index set $\mathbb{N}_0$.
- The conditions for boundedness of T are similar.
- The standard operators in applied harmonic analysis easily leads to such frames represented by bounded operators:
  - Translation: $\{T_k\varphi\}_{k\in\mathbb{Z}} = \{(T_1)^k\varphi\}_{k\in\mathbb{Z}}$;
  - Modulation: $\{E_{mb}\varphi\}_{m\in\mathbb{Z}} = \{(E_b)^m\varphi\}_{m\in\mathbb{Z}}$;
  - Scaling: $\{D_{a^j}\varphi\}_{j\in\mathbb{Z}} = \{(D_a)^j\varphi\}_{j\in\mathbb{Z}}$;

Question: Consider a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ with $ab < 1$. Does the frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ have a representation

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} = \{T^n\varphi\}_{n=-\infty}^{\infty}$$

for a bounded operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and some $\varphi \in L^2(\mathbb{R})$?

Frames of the form $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ are special!

Theorem (C., Hasannasab, Rashidi, 2017) Assume that $\{T^n \varphi\}_{n=0}^{\infty}$ is an overcomplete frame for some $\varphi \in \mathcal{H}$ and some bounded operator $T : \mathcal{H} \to \mathcal{H}$. Then the following hold:

(i) The image chain for the operator T has finite length $q(T)$.

(ii) If $N \in \mathbb{N}$, then $T^N \varphi \in \overline{\text{span}}\{T^n \varphi\}_{n=N+1}^{\infty} \Leftrightarrow N \geq q(T)$.

For any $N \geq q(T)$, let $V := \overline{\text{span}}\{T^n \varphi\}_{n=N}^{\infty}$. Then the following hold:

(iii) The space V is independent of N and has finite codimension.

(iv) The sequence $\{T^n \varphi\}_{n=N+\ell}^{\infty}$ is a frame for V for all $\ell \in \mathbb{N}_0$.

(v) V is invariant under T, and $T : V \to V$ is surjective.

(vi) If the null chain of T has finite length then $T : V \to V$ is injective; in particular this is the case if T is normal.

A way to achieve boundedness - multiple operators

Theorem (Hasannasab & C., 2016) Consider a frame $\{f_k\}_{k=1}^{\infty}$ which is norm-bounded below. Then there is a finite collection of vectors from $\{f_k\}_{k=1}^{\infty}$, to be called $\varphi_1, \dots, \varphi_J$, and corresponding bounded operators $T_j : \mathcal{H} \to \mathcal{H}$, such that

$$\{f_k\}_{k=1}^{\infty} = \bigcup_{j=1}^{J} \{T_j^n \varphi_j\}_{n=0}^{\infty}.$$

Remark: The assumption that $\{f_k\}_{k=1}^{\infty}$ is norm-bounded below can not be removed.

References

- Aceska, R., and Yeon Hyang Kim, Y. H.: *Scalability of frames generated by dynamical operators.* arXiv preprint arXiv:1608.05622 (2016).
- Aldroubi, A., Cabrelli, C., Molter, U., and Tang, S.: *Dynamical sampling.* Appl. Harm. Anal. Appl. **42** no. 3 (2017), 378-401.
- Aldroubi, A., Cabrelli, C., Çakmak, A. F., Molter, U., and Petrosyan, A.: *Iterative actions of normal operators.* J. Funct. Anal. **272** no. 3 (2017),
- Cabrelli, C., Molter, U, Paternostro, V., and Philipp, F.: *Dynamical sampling on finite index sets.* Preprint, 2017.
- Christensen, O., and Hasannasab, M.: *Frame properties of systems arising via iterative actions of operators* Preprint, 2016.
- Christensen, O., and Hasannasab, M.: *Operator representations of frames: boundedness, duality, and stability.* Int. Eq. & Op. Theory, to appear.
- Christensen, O., and Hasannasab, M.: *An open problem concerning operator representations of frames.* Preprint, 2017