Operational calculus for groups with finite propagation speed

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2 Abstract

Let A be the generator of a strongly continuous cosine family $(\cos(tA))_{t\in\mathbf{R}}$ on a complex Banach space E. The paper develops an operational calculus for integral transforms and functions of A using the generalized harmonic analysis associated to certain hypergroups. It is shown that characters of hypergroups which have Laplace representations give rise to bounded operators on E. Examples include the Mehler–Fock transform. The paper uses functional calculus for the cosine family $\cos(t\sqrt{\Delta})$ which is associated with waves that travel at unit speed. The main results include an operational calculus theorem for Sturm–Liouville hypergroups with Laplace representation as well as analogues to the Kunze–Stein phenomenon in the hypergroup convolution setting.

3 Cosine families

Let E be a separable complex Banach space and $\mathcal{L}(E)$ the algebra of bounded linear operators on E. Let A a closed and densely defined linear operator in E. Formally, a cosine family on E is a strongly continuous family $\{C(t)\}_{t \in \mathbf{R}}$ of bounded operators on E such that $C(s-t) + C(s+t) = 2C(s)C(t)$ and $C(0) = I$. Such a family admits a closed densely defined infinitesimal generator A and one naturally writes $\cos(tA)$ for $C(t)$. Cosine families arise in describing the solutions of well-posed L^2 Cauchy problems

$$\frac{\partial^2 w}{\partial t^2} = -A^2 w, \qquad w(0) = u, \qquad \frac{\partial w}{\partial t}(0) = 0$$

with initial datum $u \in L^2$. In classical situations, these systems admit wave solutions which propagate at a fixed finite speed.

Given a cosine family $\{\cos(tA)\}_{t \in \mathbf{R}}$, various authors have used this to use this to define

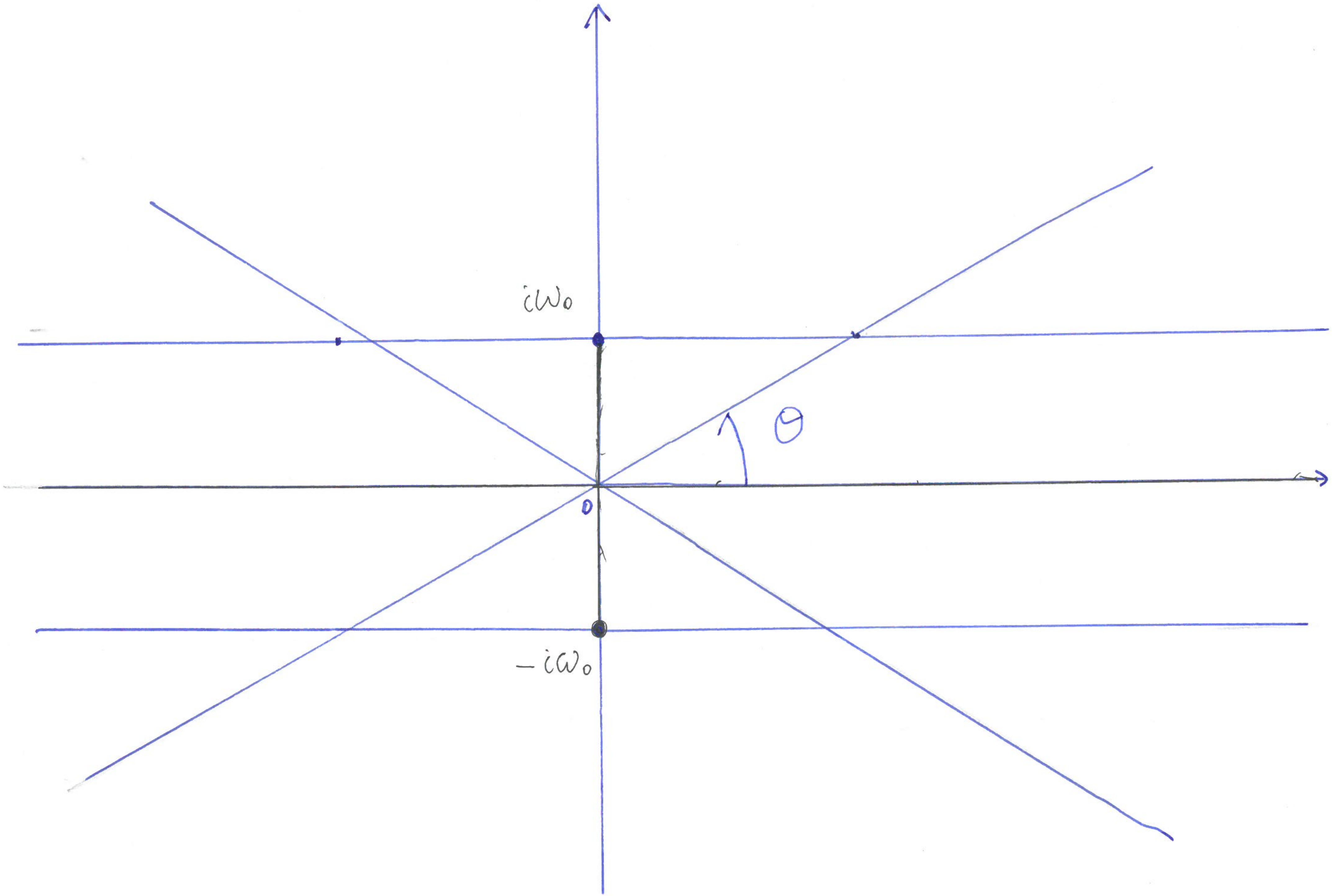
$$f(A) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F} f(t) \cos(tA)\, dt$$

4 Venturi regions

For $\omega > 0$ we let Σ_ω denote the strip $\{z \in \mathbf{C} : |\Im z| < \omega\}$ and $i\Sigma_\omega$ the corresponding vertical strip. For $0 < \theta < \pi$, we introduce the open sector $S^0_\theta = \{z \in \mathbf{C} \setminus \{0\} : |\arg z| < \theta\}$ and its reflection $-S^0_\theta = \{z : -z \in S^0_\theta\}$. An important idea is to work with holomorphic functions on 'Venturi' regions; that is, those of the form

$$V_{\theta,\omega} = \Sigma_\omega \cup S^0_\theta \cup (-S^0_\theta).$$

Likewise, $iV_{\theta,\omega}$ will denote the corresponding Venturi region with vertical axis. As usual, $H^\infty(S)$ will denote the Banach algebra of bounded analytic functions on an open subset S of the complex plane.



$i\omega_0$
θ
0
$-i\omega_0$

5 Hypergroups on $[0, \infty)$

Let $\mathbf{X} = [0, \infty)$, and $C_c(\mathbf{X})$ the space of compactly supported continuous functions $f : \mathbf{X} \to \mathbf{C}$. The set $M^b(\mathbf{X})$ of bounded Radon measures on $\mathbf{X}$ with the weak topology forms a complex vector space. When equipped with a generalized convolution' operation $M^b(\mathbf{X})$ is a convolution measure algebra called a hypergroup or 'convo' denoted $(\mathbf{X}, *)$. Denote the Dirac point mass at x by $\varepsilon_x \in M^b(\mathbf{X})$. It is a hypergroup axiom that for all $x, y \in \mathbf{X}$, $\varepsilon_x * \varepsilon_y$ is a compactly supported probability measure (generally with infinite support). The action of $*$ in a hypergroup is in fact completely determined by the convolutions $\varepsilon_x * \varepsilon_y$. When the base space is $\mathbf{X} = [0, \infty)$, the convolution $*$ is necessarily commutative, ε_0 is a multiplicative identity element. In general, hypergroups admit an involution map $x \mapsto x^-$. For $x \in \mathbf{X}$, the left translation operator Λ_x is defined, initially on $C_c(\mathbf{X})$ by

$$\Lambda_x f(y) = \int_{\mathbf{X}} f(t)\,(\varepsilon_x * \varepsilon_y)(dt) \qquad (x, y \in \mathbf{X}).$$

6 Convolution and invariant measure

It is traditional and useful to write $\Lambda_x f(y)$ as $f(x * y)$ (although this is not in fact defining an operation on X). Since $*$ is commutative, there exists an essentially unique Haar measure on $\mathbf{X}$; that is, a nontrivial positive invariant measure m on $[0, \infty)$ satisfying

$$\int_{\mathbf{X}} \Lambda_x f(y)\, m(dy) = \int_{\mathbf{X}} f(y)\, m(dy) \qquad (x \in \mathbf{X})$$

for all $f \in C_c(\mathbf{X})$. This allows us to define a (commutative) convolution between two functions $f, g \in C_c(\mathbf{X})$ by

$$(f * g)(x) = \int_{\mathbf{X}} f(y)\, \Lambda_x g(y)\, m(dy) = \int_{\mathbf{X}} f(y)\, g(x * y)\, m(dy).$$

This map extends to $L^1(m) = L^1(\mathbf{X}, m)$ and makes $(L^1(m), *)$ into a commutative Banach algebra. One often writes the convolution operation as $\Lambda_f g = f * g$ for $f, g \in L^1(m)$.

7 Multiplicative functions and characters

1. A continuous function $\phi : \mathbf{X} \to \mathbf{C}$ is said to be *multiplicative* if $\phi(x * y) = \phi(x)\phi(y)$ for all $x, y \in \mathbf{X}$ and $\phi(z) \neq 0$ for some $z \in \mathbf{X}$.
2. A *character* on the hypergroup $\mathbf{X}$ is a bounded and multiplicative function ϕ such that $\phi(x^-) = \overline{\phi(x)}$ and $\phi(0) = 1$. The *character space* $\hat{\mathbf{X}}$ is the set of all characters on $\mathbf{X}$.

When $X = [0, \infty)$ the involution is always the identity $x^- = x$, and the condition that $\phi(x^-) = \overline{\phi(x)}$ is equivalent to the condition that $\phi(x) \in \mathbf{R}$ and this simplifies some of the definitions below. The set of bounded and multiplicative functions ϕ_λ can be naturally parametrized by a domain $S_{\mathbf{X}} \subseteq \mathbf{C}$. This occurs, in particular, for Sturm–Liouville hypergroups, in which case λ is a spectral parameter.

8 The Fourier transform

The character space $\hat{\mathbf{X}}$ is always sufficiently large in our context to enable one to do harmonic analysis. We can define the Fourier transform of $f \in L^1(\mathbf{X}; m)$ by setting

$$\hat{f}(\phi) = \int_{\mathbf{X}} f(x)\phi(x)\, m(dx), \qquad (\phi \in \hat{\mathbf{X}}).$$

In the case that $\hat{\mathbf{X}} \subseteq \{\phi_\lambda : \lambda \in S_{\mathbf{X}}\}$ we shall write $\hat{f}(\lambda)$ rather than $\hat{f}(\phi_\lambda)$ and we can extend $\hat{f}$ to be a function of the complex variable λ.

9 The Plancherel measure

Theorem

(i) (Levitan) There exists a unique Plancherel measure π_0 supported on a closed subset $\mathbf{S}$ of $\hat{\mathbf{X}}$ such that $f \mapsto \hat{f}$ for $f \in L^2(m) \cap L^1(m)$ extends to a unitary isomorphism $L^2(m) \to L^2(\pi_0)$.
(ii) (Voigt) There exists a unique positive character $\phi_0 \in \mathbf{S}$, and ϕ_0 can be different from the trivial character $\mathbf{I}$.

10 Laplace representation for characters

Definition A hypergroup $(\mathbf{X}, *)$ is said to have a *Laplace representation* if $(a, b) \subseteq \mathbf{S}$ for some $0 < a < b$, and for every $x \geq 0$, there exists a positive Radon measure τ_x on $[-x, x]$ such that $\tau_x([-x, x]) = \phi_0(x)$ and for every character ϕ_λ in $\mathbf{S}$

$$\phi_\lambda(x) = \int_{-x}^{x} \cos(\lambda t)\tau_x(dt).$$

The integral is taken over $[-x, x]$, and includes any point masses at $\pm x$.

11 Extension of the Fourier transform

Lemma

Suppose that there exist $M_0, \omega_0 > 0$ such that

$$\int_{-x}^{x} \cosh(\omega_0 t)\, \tau_x(dt) \leq M_0 \qquad (x \geq 0).$$

1. Then for all $\lambda \in \Sigma_{\omega_0}$ the function $\phi_\lambda : \mathbf{X} \to \mathbf{C}$,

$$\phi_\lambda(x) = \int_{-x}^{x} \cos(\lambda t)\, \tau_x(dt) \qquad (x \geq 0)$$

   is bounded and multiplicative;
2. for all $x \in \mathbf{X}$, the map $h_x : \lambda \mapsto \phi_\lambda(x)$ is in $H^\infty(\Sigma_{\omega_0})$;
3. $\mathbf{R} \cup [-i\omega_0, i\omega_0]$ is contained in $\hat{\mathbf{X}}$;
4. the Fourier transform $f \mapsto \hat{f}$ is bounded $L^1(m) \to H^\infty(\Sigma_{\omega_0})$.

12 Laplace representation and cosine families

In this section we shall suppose that the operator A generates a strongly continuous cosine family $(\cos(tA))_{t\in\mathbf{R}}$ on E, and that $(\mathbf{X}, *)$ is a hypergroup which admits a Laplace representation for its characters ϕ_λ as given in Definition 10.
In this setting we define the family of bounded linear operators $\{\phi_A(x)\}_{x\geq 0}$ on E by the strong operator convergent integrals

$$\phi_A(x) = \int_{-x}^{x} \cos(At)\, \tau_x(dt) \qquad (x \geq 0).$$

Indeed, this enables us to deal with unbounded cosine families, as in Proposition 13 below.

13 Operational calculus for hypergroups with Laplace representation

Proposition

Let $(\mathbf{X}, *)$ have a Laplace representation satisfying and suppose that A generates a strongly continuous cosine family on E satisfying

$$\|\cos(tA)\|_{\mathcal{L}(E)} \leq \kappa \cosh(t\omega_0) \qquad (t \geq 0).$$

1. Then $(\phi_A(x))_{x>0}$ is a uniformly bounded family of operators;
2. for all $f \in L^1(m)$, the following integral converges strongly

   $$T_A(f) = \int_0^\infty f(x)\phi_A(x)\, m(dx)$$

   and defines a bounded linear operator on E;
3. for $f, g \in L^1(m)$, $T_A(f * g) = T_A(f)T_A(g)$, and so the map $T_A : L^1(m) \to \mathcal{L}(E)$ is an algebra homomorphism.

14 Kunze Stein phenomenon

For a locally compact group G, the space $L^1(G)$ acts boundedly on $L^2(G)$ by left-convolution. That is, if $f \in L^1(G)$ then $\Lambda_f : g \mapsto f * g$ is a bounded operator on $L^2(G)$. In general, this result does not extend to $f \in L^p(G)$ for $p > 1$. The Kunze–Stein phenomenon refers to the fact that for certain Lie groups, most classically for $G = \mathrm{SL}(2, \mathbf{C})$, for $1 \le p < 2$ one does obtain a bound of the form

$$\|f * g\|_{L^2(G)} \le C_p \|f\|_{L^p(G)} \|g\|_{L^2(G)}.$$

Thus the representation $\Lambda : (L^1(G), *) \to \mathcal{L}(L^2(G)) : f \mapsto \Lambda_f$ extends to a bounded linear map $\Lambda : L^p(G) \to \mathcal{L}(L^2(G))$. Indeed the classical case is $G = \mathrm{SL}(2, \mathbf{C})$ has a maximal compact subgroup $K = \mathsf{SU}(2, \mathbf{C})$ such that $K \times K$ acts upon G via $(h, k) : g \mapsto h^{-1} g k$ for $h, k \in K$ and $g \in G$, producing a space of orbits $G//K = \{KgK : g \in G\}$. The double coset space $G//K$ inherits the structure of a commutative hypergroup modelled on $\mathbf{X} = [0, \infty)$.

15 The hypergroup model of $G//K$

By results of Trimèche, there exists a commutative hypergroup on $[0,\infty)$ that has invariant measure $2^2 \sinh^2 x\, dx$. We introduce

$$\varphi_\lambda(x) = \frac{\sin \lambda x}{\lambda \sinh x} = \int_{-x}^{x} \frac{\cos \lambda t}{2 \sinh x}\, dt \qquad (\lambda \in \mathbf{C})$$

so that φ_λ is a bounded multiplicative function for $\lambda \in \Sigma_1$ and so that $\varphi_{\pm i}$ is the trivial character, so that $\omega_0 = 1$. The Plancherel measure is

$$\pi_0(d\lambda) = \frac{\lambda^2}{4\pi} \mathbf{I}_{(0,\infty)}(\lambda)\, d\lambda,$$

so that $\varphi_0(x) = x/\sinh x$ is the unique positive character in the support of π_0. Also

$$\int_0^\infty \varphi_0(x)^\nu \sinh^2 x\, dx = \int_0^\infty x^\nu \sinh^{2-\nu} x\, dx$$

converges for all $\nu > 2$.

16 Kunze–Stein phenomenon for hypergroups

Theorem

Let $(\mathbf{X}, *)$ have a Laplace representation and suppose that A generates a strongly continuous cosine family on E satisfying growth bounds as in 13. Suppose further that $\phi_0 \in L^\nu(m)$ for some $2 < \nu < \infty$. Let $0 < \alpha < 1$ and let $p = \nu/(\nu + \alpha - 1)$. Then

1. the Fourier transform $f \mapsto \hat{f}$ is bounded $L^p(m) \to H^\infty(\Sigma_{\alpha\omega_0})$;
2. the convolution operator $\Lambda_f : g \mapsto f * g$ gives a bounded linear operator on $L^2(m)$ for all $f \in L^p(m)$;
3. the map $f \mapsto T_{\alpha A}(f)$ is bounded $L^p(m) \to \mathcal{L}(E)$.

The crucial idea is that α slows the propagation speed of the hypergroup.

17 Mehler–Fock transform

1. For $\mu = 0, 1, \ldots,$ the *associated Legendre functions* may be defined to be the functions P^μ_ν such that

$$P^\mu_\nu(\cosh x) = \sqrt{\frac{2}{\pi}} \frac{(\sinh x)^\mu}{\Gamma((1/2) - \mu)} \int_0^x \frac{\cosh(\nu + (1/2))y}{(\cosh x - \cosh y)^{\mu + (1/2)}} dy.$$

2. *Legendre's functions* are defined by

$$\phi_\lambda(x) = P_{i\lambda - (1/2)}(\cosh x) = \frac{1}{\pi\sqrt{2}} \int_{-x}^x \frac{\cos \lambda y}{\sqrt{\cosh x - \cosh y}} dy.$$

An alternative notation is $R^{(0,0)}_z = P_z$ with $z = i\lambda - (1/2)$.

3. The *Mehler–Fock transform of order zero* of $f \in L^1(\sinh x\, dx)$ is

$$\hat{f}(\lambda) = \int_0^\infty f(x)\phi_\lambda(x) \sinh x\, dx.$$

18 Operations for the Mehler–Fock transform

Proposition

Let $(\cos(tA))_{t\in\mathbf{R}}$ be a cosine family on E and suppose that there exists κ such that $\|\cos(tA)\|_{\mathcal{L}(E)} \leq \kappa\cosh(t/2)$ for all $t \geq 0$. Then

1. there exists a hypergroup $([0,\infty),*)$ with Laplace representation such that $f \mapsto \hat{f}$ is the Mehler–Fock transform of order zero;
2. $(\phi_A(x))_{x>0}$ is a bounded family of operators;
3. the integral

$$T_A(f) = \int_0^\infty \phi_A(x)f(x)\sinh x\,dx \qquad (f \in L^1(\sinh x\,dx))$$

   defines a bounded linear operator such that $T_A(g*h) = T_A(g)T_A(h)$ for all $g, h \in L^1(\sinh x\,dx)$;
4. for $2 < \nu < \infty$, $0 < \alpha < 1$ and $p = \nu/(\nu+\alpha-1)$, the linear operator $f \mapsto T_{\alpha A}(f)$ is bounded $L^p(\sinh x\,dx) \to \mathcal{L}(E)$.

19 Proof

(i) Mehler showed that

$$-\phi''_\lambda(x) - \coth x\, \phi'_\lambda(x) = (\lambda^2 + (1/4))\phi_\lambda(x).$$

Trimèche introduces a hypergroup structure on $(0, \infty)$ such that the ϕ_λ for $\lambda \in \Sigma_{1/2}$ are bounded and multiplicative for this hypergroup, and he shows that the invariant measure and the Plancherel measure are supported on $[0, \infty)$, and satisfy

$$m(x)\, dx = \sinh x\, dx,$$

$$\pi_0(d\lambda) = \lambda \tanh(\pi\lambda) d\lambda,$$

so the generalized Fourier transform $\hat{f}(\lambda) = \int_0^\infty f(x)\phi_\lambda(x) m(x)\, dx$ reduces to the Mehler–Fock transform of order zero. Note that $\lambda = i/2$ gives the trivial character, which is not in the support of π_0.

20 Laplace representation and the positive character

(ii) Definition 17 gives the Laplace representation. We now observe that

$$\int_{-x}^{x} \frac{\cosh(y/2)\,dy}{\sqrt{\cosh x - \cosh y}} = \int_{-x}^{x} \frac{\cosh(y/2)\,dy}{\sqrt{\sinh^2(x/2) - \sinh^2(y/2)}}$$

is bounded, Hence Proposition 13 gives $\|\phi_A(x)\|_{\mathcal{L}(E)} \leq \kappa$.
(iii) Given that the hypergroup convolution $*$ exists, we can apply Proposition 13.
(iv) Whereas $\phi_0(x)$ can be expressed in terms of Jacobi's complete elliptic integral of the first kind with modulus $i \sinh(x/2)$, we require only the formula

$$\phi_0(x) = \frac{1}{\pi} \int_0^x \frac{dy}{\sqrt{\sinh^2(x/2) - \sinh^2(y/2)}} \leq \frac{2\sqrt{2x}}{\pi\sqrt{\sinh(x/2)}}.$$

From the differential equation we obtain $\phi_0(x) = O(xe^{-x/2})$ as $x \to \infty$, so $\phi_0 \in L^\nu(\sinh x)$ for all $2 < \nu < \infty$. Hence we can apply Theorem 16.

21 Operational calculus for Sturm–Liouville hypergroups

To produce natural examples of hypergroups as in Theorem 16, we consider certain differential operators of the form

$$L\phi(x) = -\frac{d^2\phi}{dx^2} - \frac{m'(x)}{m(x)}\frac{d\phi}{dx}, \qquad (x \geq 0).$$

Under suitable conditions on the function m, one can define a hypergroup structure on $\mathbf{X} = [0, \infty)$ for which the characters correspond to suitably normalized eigenfunctions of this operator. The Haar measure for these hypergroups is just $m(x)\,dx$ where dx is the usual Lebesgue measure on $\mathbf{X}$.

22 Conditions on Sturm–Liouville operators

Definition Suppose that $\omega_0 \geq 0$ and $\gamma > -1/2$. We say that a function $m : [0, \infty) \to [0, \infty)$ satisfies $(H(\omega_0))$ if:

1. $m(x) = x^{2\gamma+1} q(x)$ where $q \in C^\infty(\mathbf{R})$ is even, positive and $m(x)/x^{2\gamma+1} \to q(0) > 0$ as $x \to 0+$;
2. $m(x)$ increases to infinity as $x \to \infty$, and $m'(x)/m(x) \to 2\omega_0$ as $x \to \infty$; and either
3. $m'(x)/m(x)$ is decreasing; or
4. the function

$$Q(x) = \frac{1}{2}\Big(\frac{q'}{q}\Big)' + \frac{1}{4}\Big(\frac{q'}{q}\Big)^2 + \frac{2\gamma+1}{2x}\Big(\frac{q'}{q}\Big) - \omega_0^2.$$

is positive, decreasing and integrable with respect to Lebesgue measure over $(0, \infty)$.

23 Existence of Laplace representations for SL hypergroups

Lemma

Suppose that $\omega_0 > 0$ and that m satisfies $(H(\omega_0))$.

1. Then there exists a commutative hypergroup on $[0, \infty)$ such that $x^- = x$;
2. the solutions of
$$-\frac{d^2\phi_\lambda}{dx^2} - \frac{m'(x)}{m(x)}\frac{d\phi_\lambda}{dx} = (\omega_0^2 + \lambda^2)\phi_\lambda$$
such that $\phi_\lambda(0) = 1$, and $\phi'_\lambda(0) = 0$ for $\lambda \geq 0$ are characters in $\mathbf{S}$;
3. $\phi_\lambda(x)$ has a Laplace representation, where $\pm i\omega_0$ corresponds to the trivial character, and the bound holds;
4. $\hat{\mathbf{X}} = \mathbf{R} \cup [-i\omega_0, i\omega_0]$.

24 Kunze–Stein phenomenon for Sturm–Liouville hypergroups

Theorem

Suppose that m and ϕ_λ are as in Lemma 23 with $\omega_0 > 0$ and that $(\cos(tA))_{t\in\mathbf{R}}$ is a strongly continuous cosine family on E such that

$$\|\cos(tA)\|_{\mathcal{L}(E)} \le \kappa \cosh(\omega_0 t) \qquad (t \in \mathbf{R})$$

and some $\kappa < \infty$. Let $2 < \nu < \infty$, $0 < \alpha < 1$ and $p = \nu/(\nu + \alpha - 1)$.

1. Then ϕ_λ is a bounded multiplicative function on $(\mathbf{X}, *)$ for all $\lambda \in \Sigma_{\omega_0}$;
2. the Fourier transform $f \mapsto \hat{f}(\lambda)$ is bounded $L^p(m) \to H^\infty(\Sigma_{\alpha\omega_0})$;
3. $(\phi_A(x))_{x\ge 0}$ gives a bounded family of linear operators on E, $T_A(f) = \int_0^\infty f(x)\phi_A(x)m(x)\,dx$ defines a bounded linear operator on E for all $f \in L^1(m)$, and $T_A(f * g) = T_A(f)T_A(g)$ for all $f, g \in L^1(m)$;
4. the map $f \mapsto T_{\alpha A}(f)$ is bounded $L^p(m) \to \mathcal{L}(E)$.

25 Geometrical application

Let $\mathcal{M}$ be a complete Riemannian manifold of dimension n with metric ρ that has injectivity radius bounded below by some $r_0 > 0$. This means that the exponential map is injective on the tangent space above the ball $B(x, r_0) = \{y \in \mathcal{M} : \rho(x, y) \leq r_0\}$ for all $x \in \mathcal{M}$. For fixed $x_0 \in \mathcal{M}$, we can use $\rho(x, x_0)$ as the radius in a system of polar coordinates with centre x_0, noting that ρ is not differentiable on the cut locus. Let vol be the Riemannian volume measure, and for an open subset Ω with compact closure, let $\Omega_\varepsilon = \{x \in \mathcal{M} : \exists y \in \Omega : \rho(x, y) \leq \varepsilon\}$ be its ε-enlargement for $\varepsilon > 0$. Then let the outer Hausdorff measure of the boundary $\partial\Omega$ of Ω be

$$\mathrm{area}(\partial\Omega) = \limsup_{\varepsilon \to 0+} \varepsilon^{-1}(\mathrm{vol}(\Omega_\varepsilon) - \mathrm{vol}(\Omega)).$$

In particular, let $\sigma(x_0, r) = \mathrm{area}(\partial B(x_0, r))$ be the surface area of a sphere, and $m(x_0, r) = \mathrm{vol}(B(x_0, r))$ the volume of a ball.

26 Laplace operator on Riemannian manifold

The Laplace operator Δ is essentially self-adjoint on $C_c^\infty(\mathcal{M};\mathbf{C})$ by Chernoff's theorem so we can define functions of $\sqrt{\Delta}$ via the spectral theorem in $L^2(\mathcal{M},\mathrm{vol}) = L^2(\mathcal{M})$. The distributional support of $\cos t\sqrt{\Delta}\delta_{x_0}$ travels at unit speed on $\mathcal{M}$. Then for any smooth radial function $g(r)$, the Laplace operator satisfies

$$\Delta g = -g''(r) - \frac{\sigma'(x_0,r)}{\sigma(x_0,r)} g'(r).$$

For $r_0 > \delta > 0$, the modified Cheeger constant is

$$I_{\infty,\delta}(\mathcal{M}) = \inf\Big\{ \frac{\mathrm{area}(\partial\Omega)}{\mathrm{vol}(\Omega)} : \Omega \Big\}$$

where the infimum is taken over all the open subsets Ω of $\mathcal{M}$ that have compact closure, have smooth boundary $\partial\Omega$ and contain a metric ball of radius δ.

27 Radial Laplacian gives a SL hypergroup

Proposition

Let the Riemannian manifold $\mathcal{M}$ be as above and suppose that

1. $\mathcal{M}$ is noncompact with Ricci curvature bounded below by $\kappa(n-1)$ where $\kappa < 0$;
2. $I_{\infty,\delta}(\mathcal{M}) > 0$ for some $\delta > 0$;
3. $r \mapsto \log m(x_0, r)$ and $r \mapsto \log \sigma(x_0, r)$ are concave functions of $r \in (0, \infty)$.

Then $m(x_0, r)$ and $\sigma(x_0, r)$ satisfy conditions (1), (2) and (3) of Definition 22 with $2\omega_0 \geq I_{\infty,\delta}(\mathcal{M})$.

Hence the Sturm-Liouville hypergroup theory can be applied to the Laplace operator Δ on radial functions.

28 Hyperbolic space

In particular, this result applies to hyperbolic space with constant negative curvature. Let $\mathcal{H} = \{z = x + iy : y > 0\}$ and let $SL(2, \mathbf{R})$ act on $\mathcal{H}$ by linear fractional transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az + b}{cz + d}.$$

The geodesic polar coordinates with respect to centre i are (r, u), where

$$\begin{bmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{bmatrix} \begin{bmatrix} e^{-r/2} & 0 \\ 0 & e^{r/2} \end{bmatrix}$$

acting on i gives a point at distance r and angle u. The Laplace operator acting on radial functions, which depend on ρ, has eigenfunctions

$$-\phi''(r) - \coth r\, \phi'(r) = ((1/4) + t^2)\phi(r),$$

with solution

$$\phi(r) = P_{-1/2+it}(\cosh r).$$

29 Functional Calculus Problem

$$\begin{array}{ccc} f \in (L^1(m), *) & \longrightarrow & \hat{f} \in \mathcal{A} \subset H^\infty(\Sigma_\omega) \\ & \searrow & \downarrow \\ L^p(m) & \longrightarrow & \mathcal{L}(E) \end{array}$$

$$\hat{f}(A) = T_A(f) = \int f(x)\phi_A(x)m(x)dx$$

Problem

Let $V_{\theta,\omega}$ be a Venturi region that contains Σ_ω. Under what conditions on E and A is there a bounded H^∞ functional calculus map $H^\infty(V_{\theta,\omega}) \to \mathcal{L}(E)$ that extends T_A?

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