Toeplitz kernels, model spaces, and multipliers

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Hardy spaces

As usual $H^2(\mathbb{D})$ is the Hardy space of the unit disc $\mathbb{D}$, the functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

It embeds isometrically as a subspace of $L^2(\mathbb{T})$, with $\mathbb{T}$ the unit circle,

$$f(e^{it}) \sim \sum_{n=0}^{\infty} a_n e^{int}.$$

Orthogonal decomposition

Indeed we may write

$$L^2(\mathbb{T}) = H^2 \oplus (H^2)^\perp,$$

so that

$$\sum_{n=-\infty}^{\infty} a_n e^{int} = \sum_{n=0}^{\infty} a_n e^{int} + \sum_{n=-\infty}^{-1} a_n e^{int},$$

and

$$f \in H^2 \iff \bar{z}\bar{f} \in (H^2)^\perp = \overline{H_0^2}.$$

Here, and usually from now on, we write $z = e^{it}$.

Toeplitz operators in brief

For $g \in L^\infty(\mathbb{T})$ we define the Toeplitz operator T_g on H^2 by

$$T_g f = P_{H^2}(gf) \qquad (f \in H^2),$$

or multiplication followed by orthogonal projection. It is well known that $\|T_g\| = \|g\|_\infty$, and if g has Fourier coefficients (c_n), then T_g has the matrix

$$\begin{pmatrix} c_0 & c_{-1} & c_{-2} & \cdots \\ c_1 & c_0 & c_{-1} & \ddots \\ c_2 & c_1 & c_0 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Inner–outer factorizations

Recall that if $f \in H^2$, not the 0 function, then it has an inner–outer factorization (unique up to unimodular constants)

$$f = \theta u$$

with θ inner, i.e., $|\theta(e^{it})| = 1$ a.e., and with u outer (no nontrivial inner divisors). Equivalently,

$$\overline{\mathrm{span}}(u, zu, z^2 u, \ldots) = H^2.$$

Model spaces

The factorization follows from Beurling's theorem, that the non-trivial closed invariant subspaces for the shift $S = T_z$ are the subspaces θH^2, with θ inner.

Now it follows that the invariant subspaces for the backwards shift $S^* = T_{\overline{z}}$ are the **model spaces**

$$K_\theta = H^2 \ominus \theta H^2 = H^2 \cap \theta \overline{H_0^2}$$

with θ inner.

It is easy to check that $K_\theta = \ker T_{\overline{\theta}}$.

Examples

(i) Take $\theta(z) = z^n$, and then

$$K_\theta = \mathrm{span}(1, z, z^2, \ldots, z^{n-1}).$$

(ii) Take

$$\theta(z) = \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z},$$

a finite Blaschke product with distinct zeroes $a_1, a_2, \ldots, a_n$ in $\mathbb{D}$. Then

$$K_\theta = \mathrm{span}\left(\frac{1}{1 - \overline{a_1} z}, \ldots, \frac{1}{1 - \overline{a_n} z} \right).$$

On the right half-plane $\mathbb{C}_+$

For an infinite-dimensional example, let $\mathcal{L}$ denote the Laplace transform, and consider, for $T > 0$,

$$\begin{array}{ccc} L^2(0,T) & \hookrightarrow & L^2(0,\infty) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ K_\theta & \hookrightarrow & H^2(\mathbb{C}_+) \end{array}$$

with $H^2(\mathbb{C}_+)$ the Hardy space on $\mathbb{C}_+$.

Here $\mathcal{L}$ acts as an isomorphism, θ is the inner function $\theta(s) = e^{-sT}$, and K_θ is its model space (a Paley–Wiener space).

General Toeplitz kernels

We mentioned that $K_\theta = \ker T_{\overline{\theta}}$, so let's look at general Toeplitz kernels (T-kernels for short), $\ker T_g$.

These are **nearly invariant**, i.e., if $f \in \ker T_g$, with $f(0) = 0$, then $f(z)/z$ is in $\ker T_g$.

Proof: $gf \in \overline{H_0^2}$ and so $\overline{z}gf \in \overline{H_0^2}$.

In fact Toeplitz kernels have the stronger property that one can divide out any inner function θ, not just $\theta(z) = z$.

Hitt (1988) and then Sarason (also 1988) classified nearly-invariant subspaces.

Nearly-invariant subspaces

Near invariance means that if $f \in N$, with $f(0) = 0$, then $f(z)/z$ is in N.

They have the form $N = FK$, where K is a model space, $\{0\}$, or H^2, and F is an isometric multiplier. That is,

$$\|Fh\| = \|h\| \qquad (h \in K).$$

For Toeplitz kernels (apart from $\{0\}$, which we'll always exclude), F will actually be an outer function, and K will be a K_θ.

Minimal Toeplitz kernels

Let $f \in H^2$. Then there is a minimal T-kernel K containing f. That is, $f \in K = \ker T_g$ for some $g \in L^\infty$, and if $f \in \ker T_h$ then $\ker T_g \subseteq \ker T_h$.

We write $K = \mathcal{K}_{\min}(f)$.

Indeed if $f = \theta u$ (inner/outer factorization), then we may take

$$g = \frac{\overline{z}\overline{\theta}\overline{u}}{u},$$

(Câmara–JRP, 2014, using ideas of Sarason et al).

Note that g is even unimodular.

The vectorial case

For $n \geq 1$, let $(H^2)^n = H^2(\mathbb{D}, \mathbb{C}^n)$ denote the Hardy space of $\mathbb{C}^n$-valued functions, with the obvious Hilbert space norm.

We can make a similar definition of Toeplitz operators $(H^2)^n \to (H^2)^n$, with matrix-valued symbols in $L^{\infty, n \times n}$.

It is still unknown whether every function in $(H^2)^n$ is contained in a minimal T-kernel.

In the rational case, the result does hold.

Maximal vectors

Back in the scalar case, it now turns out that Toeplitz kernels are not so complicated after all.

Theorem. [CP14] Every Toeplitz kernel K is $\mathcal{K}_{\min}(f)$ for some $f \in K$.

Indeed, if $K \neq \{0\}$ and $K = \ker T_g$, then $K = \mathcal{K}_{\min}(k)$ if and only if $k \in H^2$ and

$$k = g^{-1}\overline{zu},$$

with u outer. We call these **maximal vectors**.

Example

With $\theta(z) = z^2$, $K_\theta = \mathrm{span}(1, z)$.

The maximal vectors are $k(z) = a + bz$ where $\overline{a}z + \overline{b}$ is outer. That is, $0 \le |a| \le |b|$, $b \neq 0$.

For example, $\mathcal{K}_{\min}(z + \frac{1}{2}) = K_\theta$, as by near-invariance we can divide out the inner function $(z + \frac{1}{2})/(1 + z/2)$, and so $1 + z/2$ is also in the minimal kernel.

But $\mathcal{K}_{\min}(z + 2) = \mathrm{span}(z + 2)$, a Toeplitz kernel but not a model space.

Indeed, $\mathrm{span}(z + 2) = \ker T_g$, with

$$g(z) = \frac{\overline{z}(\overline{z} + 2)}{z + 2}.$$

Multiliers

For arbitrary subspaces $P, Q \subseteq H^2$, we write

$$M(P, Q) = \{w \in \mathrm{Hol}(\mathbb{D}) : wp \in Q \text{ for all } p \in P\}.$$

If P is a Toeplitz kernel, then it contains an outer function (these have no zeroes) so all multipliers from P to Q are automatically in $\mathrm{Hol}(\mathbb{D})$.

Functions in $M(P, Q)$ need not lie in H^2, although they do if P is a model space.

Multipliers again

For example,

$$f(z) = (z-1)^{1/2}$$

spans a 1-dimensional T-kernel $K = T_g$ with symbol

$$g(z) = z^{-3/2}$$

with $\arg z \in [0, 2\pi)$ on $\mathbb{T}$. (Trust me...)

So the function $1/f$, which is not in H^2, multiplies K onto the space of constant functions, which is $\ker T_{\overline{z}}$.

Multipliers for model spaces

Theorem: (Fricain, Hartmann, Ross, 2016). For θ, φ inner, $w \in M(K_\theta, K_\varphi)$ if and only if both

(i) $w(S^*\theta) \in K_\varphi$ (note that $(S^*\theta)(z) = \frac{\theta(z)-\theta(0)}{z}$), and

(ii) $wK_\theta \subset L^2(\mathbb{T})$.

The second condition says that $|w|^2 dm$ is a Carleson measure for K_θ.

Since $S^*\theta \in K_\theta$ both conditions are obviously necessary.

Multipliers for Toeplitz kernels

One generalization (Câmara–JRP, 2016) goes as follows:

Assume that $\ker T_g$ and $\ker T_h$ are non-trivial. Then $w \in M(\ker T_g, \ker T_h)$ if and only if both

(i) $wk \in \ker T_h$ for some (and hence every) maximal vector k in $\ker T_g$, and

(ii) $w \ker T_g \subseteq L^2(\mathbb{T})$.

Again both conditions are obviously necessary, and since $K_\theta = \mathcal{K}_{\min}(S^*\theta)$ for θ inner, we may deduce the [FHR] result.

What test functions can we use?

In fact **only** maximal vectors can be used.

For if $k \in \ker T_g$ and suppose that k is not maximal, i.e., $\mathcal{K}_{\min}(k) \subsetneq \ker T_g$.

Then the multiplier $w(z) = 1$ maps k into $\mathcal{K}_{\min}(k)$, but doesn't map $\ker T_g$ into $\mathcal{K}_{\min}(k)$.

Often, reproducing kernels are used as test functions (e.g. for boundedness of Hankel operators and Carleson measures), but not here, since in general they are not maximal.

Carleson measures for T-kernels

Since we have $\ker T_g = FK_\theta$ for some inner function θ and some isometric multiplier F, the Carleson measures for $\ker T_g$ can be expressed in terms of those for K_θ.

A partial classification for K_θ (fairly transparent, but only for some inner functions) is given by Cohn (1982).

A full classification (less transparent) is in a recent preprint of Lacey, Sawyer, Shen, Uriarte-Tuero and Wick (2017).

Surjective multipliers

Crofoot (1994) looked at surjective multipliers for model spaces, i.e., $wK_\theta = K_\varphi$.
These exist only when θ and φ are related by a disc automorphism, i.e, $\varphi = \tau \circ \theta$.

For T-kernels we have:

Theorem: $w \ker T_g = \ker T_h$ if and only if $w \ker T_g \subset L^2(\mathbb{T})$, $w^{-1} \ker T_h \subset L^2(\mathbb{T})$, and

$$h = g \frac{\overline{w}}{w} \frac{\overline{v}}{\overline{u}},$$

with u, v outer.
For model spaces this leads quickly to Crofoot's result.

The right half-plane

In the L^2 case there is a unitary equivalence between Hardy spaces on the disc and half-plane that preserves Toeplitz kernels.

Thus, we have analogous results, e.g.,

$w \in M(\ker T_g, \ker T_h)$ if and only if

(i) $wk \in \ker T_h$ for some (and hence every) maximal vector k in $\ker T_g$, and

(ii) $w \ker T_g \subseteq L^2(i\mathbb{R})$.

A suitable choice for K_θ is $k(s) = \dfrac{\theta(s) - \theta(1)}{s - 1}$.

What is different about the half-plane?

Note that on the half-plane K_θ can be infinite-dimensional but still contained in H^∞ (not possible on the disc).

Thus for $w \in H^2$ the Carleson measure condition is automatically satisfied.

In particular, this happens for $\theta(s) = e^{-sT}$, giving the Paley–Wiener model spaces.

There are applications in finite-time convolution operators (which correspond to multipliers by the inverse Laplace transform).

Related work

Closely related to multipliers are **truncated Toeplitz operators** $A_g^{\theta,\varphi}$ mapping K_θ to K_φ by

$$A_g^{\theta,\varphi} f = P_{K_\varphi}(gf) \qquad (f \in K_\theta).$$

In the case of bounded g these are equivalent after extension to Toeplitz operators on $(H^2)^2$ with 2×2 matrix-valued symbols.

That's all. Thank you.