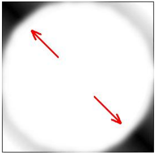
Limit Operators

Getting your hands on the essentials.



Marko Lindner

TUHH

Technische Universität Hamburg

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IWOTA Chemnitz 2017

...with the help of...

This talk is based on joint work with

- Markus Seidel, Zwickau
- Raffael Hagger, Hannover
- Hagen Söding, TU Hamburg
- Simon Chandler-Wilde, Reading
- Bernd Silbermann, Chemnitz

For a bounded linear operator

$$\mathcal{A} \;:\; X \to X$$

on a Banach space X, choose a basis in X and represent $\mathcal{A}$ as an **infinite matrix**.

Sometimes it is convenient to number the basis elements over the integers $\mathbb{Z}$ (rather than the naturals $\mathbb{N}$), leading to a **bi-infinite matrix**:

$$A = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & a_{-1,-1} & a_{-1,0} & a_{-1,1} & \cdots \\ \cdots & a_{0,-1} & a_{0,0} & a_{0,1} & \cdots \\ \cdots & a_{1,-1} & a_{1,0} & a_{1,1} & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We will mostly think of **banded matrices** A with uniformly bounded entries: $\sup |a_{ij}| < \infty$, so A acts as a bounded linear operator on $\ell^p(\mathbb{Z})$, $p \in [1, \infty]$.

For a Banach space X, put

$$\mathcal{L}(X) = \text{the set (Banach algebra) of all } \textbf{bounded} \text{ linear operators } X \to X,$$
$$\mathcal{K}(X) = \text{the set of all } \textbf{compact} \text{ operators } X \to X \text{ (closed ideal in } \mathcal{L}(X)).$$

Then one can form the factor algebra

The Calkin algebra

$$\mathcal{L}(X)/\mathcal{K}(X) = \{A + \mathcal{K}(X) : A \in \mathcal{L}(X)\}.$$

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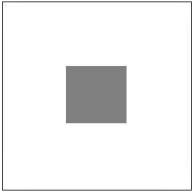
$$\mathcal{L}(X)/\mathcal{K}(X) = \{A + \mathcal{K}(X) : A \in \mathcal{L}(X)\}.$$

More specifically, for $X = \ell^p(\mathbb{Z})$, let

$$\mathrm{BO}(X) = \text{the set (algebra) of all operators } X \to X \text{ with a } \textbf{band matrix},$$
$$\mathrm{BDO}(X) = \text{the norm closure (Banach algebra) of } \mathrm{BO}(X).$$

Operator classes graphically



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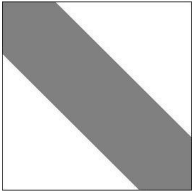

$= \mathrm{BO}(X)$

band operators

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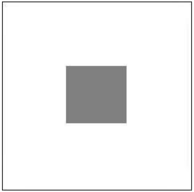
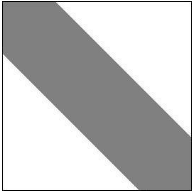
 $\subset$  $= \mathrm{BO}(X)$
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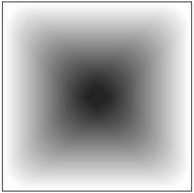
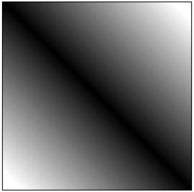
 $\subset$  $= \mathrm{BDO}(X)$
band-dominated operators

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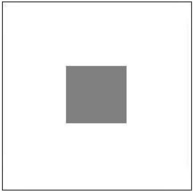

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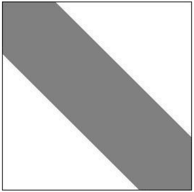

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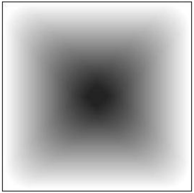

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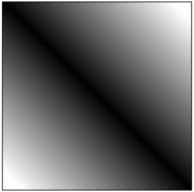

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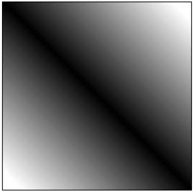
$\mathrm{BDO}(X)/\mathcal{K}(X) =$



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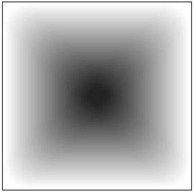

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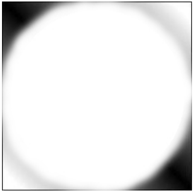

Let $A \in \mathcal{L}(X)$.

Definition: essential norm

$$\|A\|_{\mathrm{ess}} \;:=\; \|A + \mathcal{K}(X)\| \;=\; \inf\{\|A + K\| : K \in \mathcal{K}(X)\} \;=\; \operatorname{dist}(A, \mathcal{K}(X))$$

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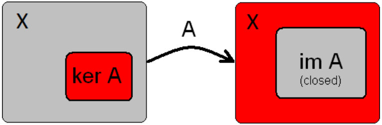
Definition: Fredholmness

A is Fredholm ("essentially invertible") iff $A + \mathcal{K}(X)$ is invertible in $\mathcal{L}(X)/\mathcal{K}(X)$.

A is Fredholm iff its kernel and cokernel have finite dimension.



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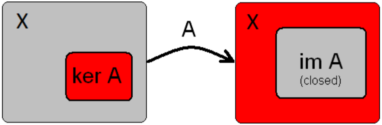

Definition: Essential spectrum

$$\operatorname{spec}_{\text{ess}} A \;:=\; \operatorname{spec}_{\mathcal{L}(X)/\mathcal{K}(X)}(A + \mathcal{K}(X)) \;=\; \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}$$

Essential spectrum and norm: diagonal examples

For

$$A = \begin{pmatrix} \ddots & & & & & & & \\ & 3 & & & & & & \\ & & 5 & & & & & \\ & & & 3 & & & & \\ & & & & 5 & & & \\ & & & & & 3 & & \\ & & & & & & 3 & \\ & & & & & & & \ddots \end{pmatrix},$$

we clearly have

$$\operatorname{spec} A = \{3, 5\}, \qquad \|A\| = 5$$

but

$$\operatorname{spec}_{\text{ess}} A = \{3\}, \qquad \|A\|_{\text{ess}} = 3.$$

For

$$A = \begin{pmatrix} \ddots & & & & & & \\ & 3 & & & & & \\ & & \color{red}{5} & & & & \\ & & & 3 & & & \\ & & & & \color{red}{5} & & \\ & & & & & 3 & \\ & & & & & & 3 \\ & & & & & & & \ddots \end{pmatrix},$$

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The spectral value $\color{red}{5}$ is not essential ("not visible at ∞").
$A - \color{red}{5}I$ is not invertible but still Fredholm (kernel and cokernel have finite dimension).

Essential spectrum and norm: diagonal examples

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$$A = \begin{pmatrix} \ddots & & & & & \\ & 3+\varepsilon_{-1} & & & & \\ & & 3+\varepsilon_0 & & & \\ & & & 3+\varepsilon_1 & & \\ & & & & 3+\varepsilon_2 & \\ & & & & & \ddots \end{pmatrix},$$

with positive ε_n such that $\varepsilon_n \to 0$ as $n \to \pm\infty$, we have

$$\operatorname{spec} A = \{3+\varepsilon_n : n \in \mathbb{Z}\} \cup \{3\}, \qquad \|A\| = \max\{3+\varepsilon_n : n \in \mathbb{Z}\}.$$

The spectral value 3 is no eigenvalue but still in the spectrum.
$A - 3I$ is injective but has no bounded inverse (not Fredholm, range not closed).

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$$\operatorname{spec} A = \{3+\varepsilon_n, 5+\varepsilon_n : n \in \mathbb{Z}\} \cup \{3,5\}, \qquad \|A\| = \max\{3+\varepsilon_n, 5+\varepsilon_n : n \in \mathbb{Z}\}.$$

The spectral values 3 and 5 are no eigenvalues of A.
$A - 3I$ and $A - 5I$ are injective but have no bounded inverse (range not closed).

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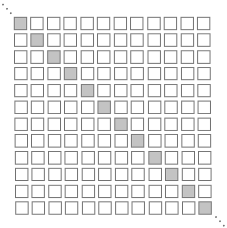
It holds

$$\operatorname{spec}_{\text{ess}} A = \{3,5\}, \qquad \|A\|_{\text{ess}} = 5.$$

For a general (bounded) **diagonal** matrix

$$A = \begin{pmatrix} \ddots & & & & & \\ & a_{-1,-1} & & & & \\ & & a_{0,0} & & & \\ & & & a_{1,1} & & \\ & & & & a_{2,2} & \\ & & & & & \ddots \end{pmatrix},$$



it holds that

$$\text{spec}_{\text{ess}} A \;=\; \text{the set of all partial limits of the sequence } (a_{n,n})_{n\in\mathbb{Z}}.$$

In other words:

$$\lambda \in \text{spec}_{\text{ess}} A \quad\Longleftrightarrow\quad \exists n_1, n_2, \dots \to \pm\infty :\; a_{n_k,n_k} \to \lambda.$$

Moreover,

$$\|A\|_{\text{ess}} \;=\; \text{the largest (in modulus) partial limit} \;=\; \limsup |a_{n,n}|.$$

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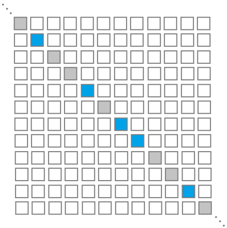

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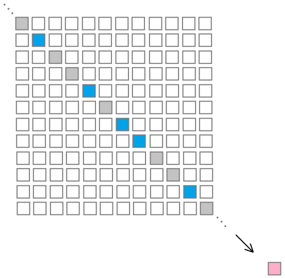

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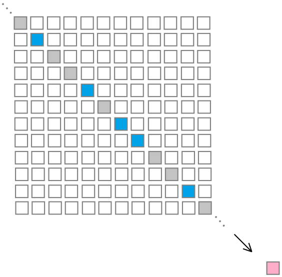

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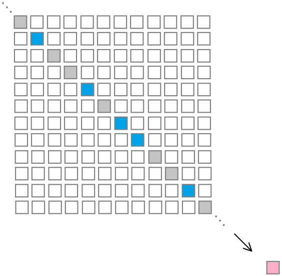

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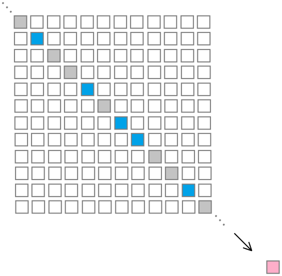

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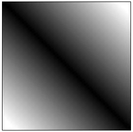
The whole coset $A + \mathcal{K}(X) \in \mathcal{L}(X)/\mathcal{K}(X)$ is encoded in the partial limits of $(a_{n,n})_{n\in\mathbb{Z}}$. Restricting consideration to diagonal matrices, the Calkin algebra is

$$\mathcal{L}_{\text{diag}}(X)/\mathcal{K}_{\text{diag}}(X) \;\cong\; \ell^\infty/c_0.$$

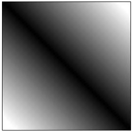
From **diagonal** to **band-dominated** matrices



For $A \in \mathrm{BDO}(X)$, the coset $A + \mathcal{K}(X)$ is still determined by the asymptotics of $A = (a_{i,j})$ at infinity. Again, take a sequence $n_1, n_2, \dots \to \pm\infty$ and

$$\text{follow the entries } a_{n_k,n_k} \text{ as } k \to \infty. \tag{1}$$

New: Now also the **context** of the entries (1) is important.



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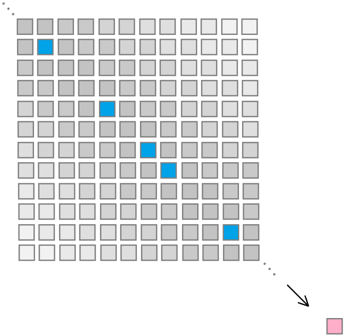
New: Now also the **context** of the entries (1) is important.

Not only the sequence (1) itself shall converge but also its neighbour entries:

$$a_{n_k+i,n_k+j} \to: b_{i,j} \qquad \forall i,j \in \mathbb{Z}.$$

The existence of such sequences $h = (n_k)$ is guaranteed by the Bolzano-Weierstrass theorem.



Definition: limit operator

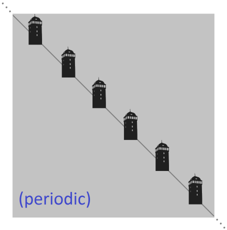
The operator with matrix $B = (b_{i,j})_{i,j\in\mathbb{Z}}$ is called *limit operator of A w.r.t. the sequence h.*

We write A_h for B and $\sigma^{\mathrm{op}}(A)$ for the set of all A_h.

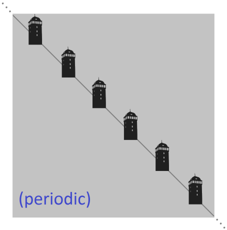
Limit operators: Time for examples

A periodic matrix:

$\sigma^{\text{op}}\Big($



$\Big) = \Big\{$ all shifts of



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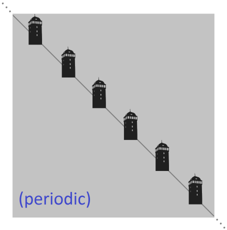
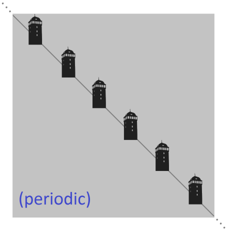
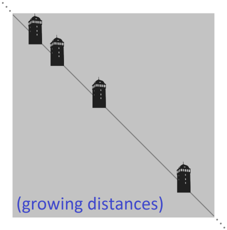
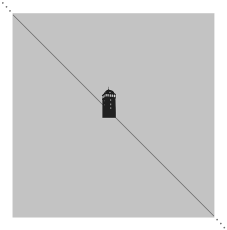
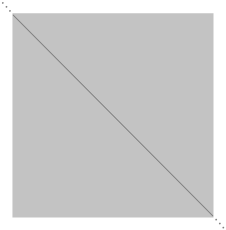
$\sigma^{\mathrm{op}}\Big($ 

$\Big) = \Big\{$all shifts of 

$\Big\}$

Simple but non-periodic:

$\sigma^{\mathrm{op}}\Big($ 

$\Big) = \Big\{$all shifts of  ,  $\Big\}$

Discrete Schrödinger operator in 1D

$$(Ax)_n \;=\; x_{n-1} + v(n)x_n + x_{n+1}, \qquad n \in \mathbb{Z}$$

with a bounded potential $v \in \ell^\infty(\mathbb{Z})$. The matrix looks like this

$$A \;=\; \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & v_{-2} & 1 & & & & \\ & 1 & v_{-1} & 1 & & & \\ & & 1 & v_0 & 1 & & \\ & & & 1 & v_1 & 1 & \\ & & & & 1 & v_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}$$

Limit operators: Time for examples

Discrete Schrödinger operator in 1D

$$(Ax)_n \;=\; x_{n-1} + v(n)x_n + x_{n+1}, \qquad n \in \mathbb{Z}$$

with a bounded potential $v \in \ell^\infty(\mathbb{Z})$. The matrix looks like this

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Limit op's of A:

$$(Bx)_n \;=\; x_{n-1} + w(n)x_n + x_{n+1}, \qquad n \in \mathbb{Z}$$

with a potential w "locally representing v at infinity".

So it is enough to look at the potential v:

Example 1: locally constant potential

$$v = (\cdots, \underbrace{\beta,\beta,\beta,\beta}_{4}, \underbrace{\alpha,\alpha,\alpha}_{3}, \underbrace{\beta,\beta}_{2}, \underbrace{\alpha}_{1}, \underbrace{\beta,\beta}_{2}, \underbrace{\alpha,\alpha,\alpha}_{3}, \underbrace{\beta,\beta,\beta,\beta}_{4}, \cdots), \qquad \alpha \neq \beta$$

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$\Rightarrow$ 4 limop's:

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$$v(n+1) - v(n) \ \to \ 0, \qquad n \to \infty$$

e.g. $v(n) = \cos\sqrt{|n|}$.

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$\Rightarrow$ Limop's (all **constant**):

$$w(n) \equiv a, \qquad a \in v(\infty)$$

Example 3: random (actually pseudo-ergodic) potential

Take random (i.i.d.) samples $v(n)$ from a compact set $V \subset \mathbb{C}$.

$\stackrel{a.s.}{\Longrightarrow}$ The infinite "word" $(\cdots, v(-1), v(0), v(1), \cdots)$ contains **every** finite word over V as a subword (up to arbitrary accuracy $\varepsilon > 0$). [*pseudo-ergodic*, Davies 2001]

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$$v(n) \;=\; \cos(n\alpha), \qquad n \in \mathbb{Z}$$

Case 1: $\alpha = \frac{p}{q} 2\pi \in \pi\mathbb{Q}$ (periodic)
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Case 2: $\alpha \notin \pi\mathbb{Q}$ (almost-periodic, see Almost-Mathieu operator)
$\Rightarrow$ ∞-many limop's:

$$w_\theta(n) = \cos(n\alpha + \theta), \qquad \theta \in [0, 2\pi)$$

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$(S_{-n}AS_n)_{n\in\mathbb{Z}^d}$ or, likewise, of the function $f_A : n \in \mathbb{Z}^d \mapsto S_{-n}AS_n \in \mathrm{BDO}(X)$.

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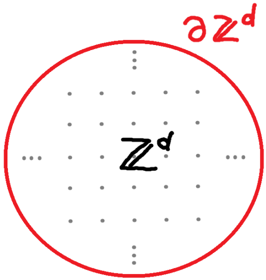
- Evaluate f_A at the **boundary** $\partial\mathbb{Z}^d$

$\Rightarrow$ limit operators of A



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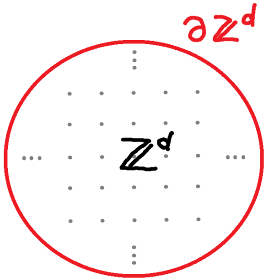

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These ideas can be extended from $\ell^p(\mathbb{Z}^d)$ [Rabinovich, Roch, Silbermann 1998] to

- $\ell^p(G)$ for finitely generated discrete groups G [Roe 2005]
- $\ell^p(X)$ for strongly discrete metric spaces X [Spakula & Willett 2014]
- $L^p(X,\mu)$ for fairly general metric spaces X and measures μ [Hagger & Seifert 2017_{+x}]

So we have

New enumeration (independent of A) of the limit operators of A

$$\begin{aligned}\sigma^{\mathrm{op}}(A) &= \{A_h : h = (n_1, n_2, \ldots) \text{ in } \mathbb{Z}^d \text{ with } |n_k| \to \infty \text{ s.t. } \lim S_{-n_k} A S_{n_k} \text{ exists}\} \\ &= \{A_g : g \in \partial\mathbb{Z}^d\}\end{aligned}$$

Now one can add or multiply two instances of $\sigma^{\mathrm{op}}(A)$ elementwise and get

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Limit operators: The definition revisited

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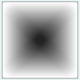
Key observation

The kernel of that homomorphism $\varphi : A \mapsto \sigma^{\mathrm{op}}(A)$ is $K(X)$.
So $A + K(X) \mapsto \sigma^{\mathrm{op}}(A)$ is an **isomorphism** $\mathrm{BDO}(X)/\ker\varphi \to \mathrm{im}\,\varphi$.



Limit operators and our essentials

The result is an identification

$$A + \mathcal{K}(X) \quad \cong \quad \sigma^{\mathrm{op}}(A)$$

for all $A \in \mathrm{BDO}(X)$ – with the following consequences:

A	$A + \mathcal{K}(X)$	$\sigma^{\mathrm{op}}(A)$
essential norm $\|A\|_{\mathrm{ess}}$	$\|\mathcal{A} + \mathcal{K}(X)\|_{\mathcal{L}(X)/\mathcal{K}(X)}$	$\max_h \|A_h\|$

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A is Fredholm	all A_h are invertible	[Lange, Rabinovich 1985+] [Rabinovich, Roch, Silbermann 1998+] [ML, Silbermann 2003], [ML 2003+] [Chandler-Wilde, ML 2007] [Seidel, ML 2014]
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Self-similar operators

Definition: self-similar operator

We say that $A \in \mathrm{BDO}(X)$ is *self-similar* if $A \in \sigma^{\mathrm{op}}(A)$.

Roughly speaking, this means that

- A contains a copy of itself, at infinity.
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"essential stuff = real stuff."

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$$Ax = b.$$

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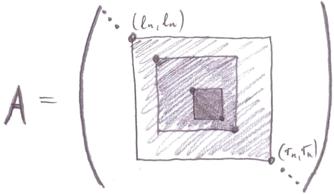

and, assuming that A is invertible, hope

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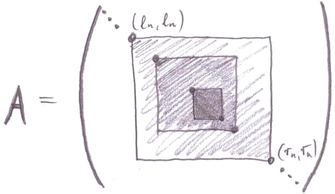

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It turns out: This "hope" will come true iff the sequence (A_n) is **stable**, meaning that

all A_n with sufficiently large n are invertible and $\sup_{n \geq n_0} \|A_n^{-1}\| < \infty$.

Stability is just another "essential"

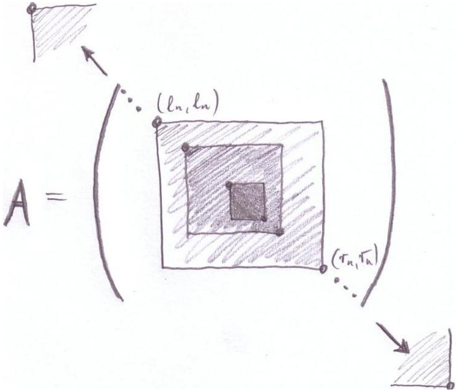
The sequence (A_n) is **stable** $\iff$ $D := Diag(A_1, A_2, \ldots)$ is **Fredholm**.

This brings us back to limit operators of D – and hence of A.

In the end we have to follow the two "corners" (semi-infinite matrices)

$$\begin{pmatrix} a_{l_n,l_n} & \cdots \\ \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \ddots & \vdots \\ \cdots & a_{r_n,r_n} \end{pmatrix}$$

of A_n as $n \to \infty$ and find (partial) limits of these matrix sequences:



Following the corners as they move out to infinity

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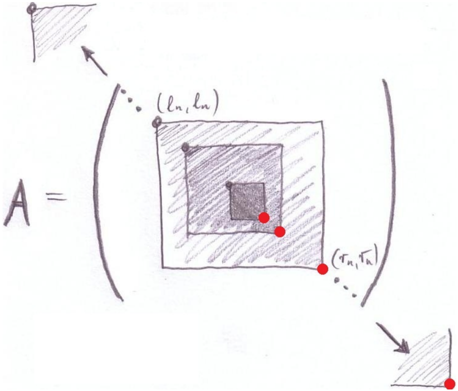

Following the corners as they move out to infinity

In the end we have to follow the two "corners" (semi-infinite matrices)

$$\begin{pmatrix} a_{l_n,l_n} & \cdots \\ \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \ddots & \vdots \\ \cdots & a_{r_n,r_n} \end{pmatrix}$$

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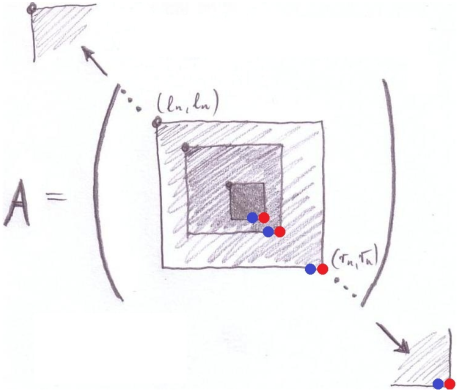

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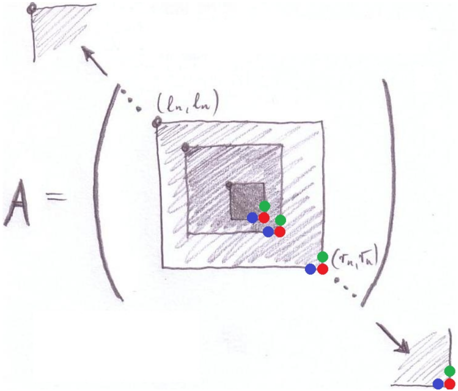

Table of Contents

The Fibonacci Hamiltonian is a particular discrete Schrödinger operator in 1D:

$$(Ax)_n = x_{n-1} + v_n x_n + x_{n+1}, \qquad n \in \mathbb{Z}.$$

So, again, the matrix looks like this

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & v_{-2} & 1 & & & & \\ & 1 & v_{-1} & 1 & & & \\ & & 1 & v_0 & 1 & & \\ & & & 1 & v_1 & 1 & \\ & & & & 1 & v_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}.$$

The potential v only assumes the values 0 and 1 – but in a very interesting pattern.

50 letters of the Fibonacci word ("quasiperiodic")

$$\ldots 10110101101101011010110110101101101011010110110101 \ldots$$

Fibonacci and his rabbit population

time	population	count
1		
2		
3		
4		
5		
6		
7		

time	population	count
1	1	1
2	10	2
3	101	3
4	10110	5
5	10110101	8
6	1011010110110	13
7	101101011011010110101	21
8	1011010110110101101011011010110110	34
9	1011010110110101101011011010110110101101011011010110101	55
⋮	⋮	⋮

Three equivalent constructions of the Fibonacci word

$$0 \mapsto 1, 1 \mapsto 10; \qquad f_{k+1} := f_k f_{k-1}; \qquad v_n = \chi_{[1-\alpha,1)}(n\alpha \bmod 1), \;\; \alpha = \frac{2}{1+\sqrt{5}}$$

The last formula is also used to define v_n for all $n \in \mathbb{Z}$. ($\Rightarrow$ bi-infinite Fibonacci word)

The “mod 1” rotation formula and limit operators

$$v_n \;=\; \chi_{[1-\alpha,1)}(\;\; n\alpha \;\;\; \text{mod } 1\;), \qquad n\in\mathbb{Z}, \quad \alpha = \frac{2}{1+\sqrt{5}} \quad (\text{“golden mean”})$$

n	...	-3	-2	-1	0	1	2	3	4	5	6	7	8	$\cdots$
v_n														

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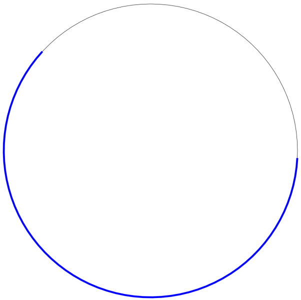

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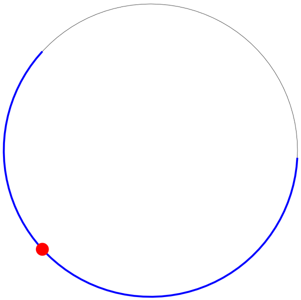

n	...	-3	-2	-1	0	1	2	3	4	5	6	7	8	$\cdots$
v_n						1								

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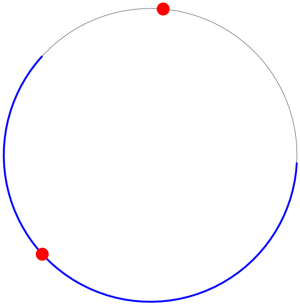

n	...	-3	-2	-1	0	1	2	3	4	5	6	7	8	$\cdots$
v_n						1	0							

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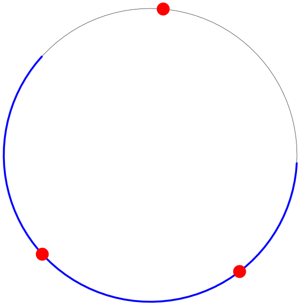

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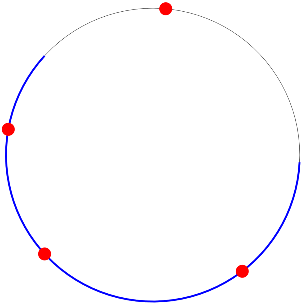

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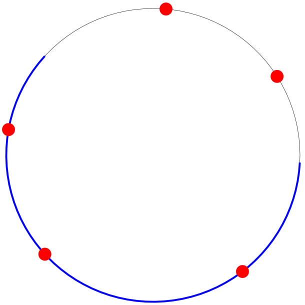

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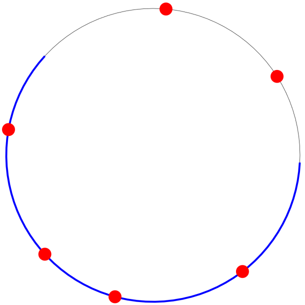

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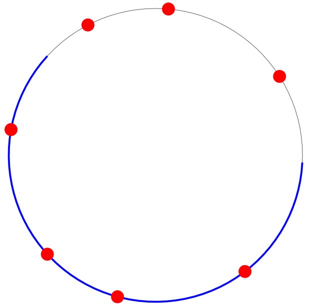

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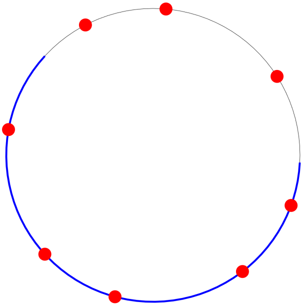

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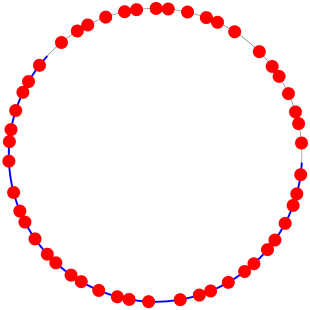

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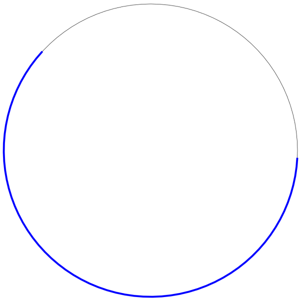

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In an infinite random word over the alphabet $\{0, 1\}$ you can find (almost surely) all 2^n subwords of length n.

In contrast: How many can you find in the Fibonacci word

$$v = \cdots 10110101101101011010110110101101101011010110110101 \cdots ?$$

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length	subwords	count
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⋮		⋮
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Interesting feature: Very moderate (in fact: minimal) growth, compared to 2^n.

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Interesting feature: Very moderate (in fact: minimal) growth, compared to 2^n.

One can show:
The main diagonal of **every** limit operator of A has **the same** list of subwords!

Let

- $v = \cdots 1011010110110101101011011010110110101 1\cdots$ be the Fibonacci word,
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The main diagonal of the limit operator A_h has the same subwords as that of A.

$$w \prec v \quad \Longleftrightarrow \quad w \prec v_h$$

Limit operators and their subwords

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$\boxed{\Leftarrow}$ $w \prec v_h \quad \Longrightarrow \quad w \prec S_{-n_k} v$ for large $k \quad \Longrightarrow \quad w \prec v.$

$\boxed{\Rightarrow}$ Let $w \prec v$, say (w.l.o.g.) $w \prec v_+$.

$\Longrightarrow$ Every $S_{-n_k} v$ contains w in a F_{n+1}-neighbourhood of zero.

$\Longrightarrow$ Every limit potential v_h contains w in a F_{n+1}-neighbourhood of zero.

Limit operators and the "mod 1" rotation formula

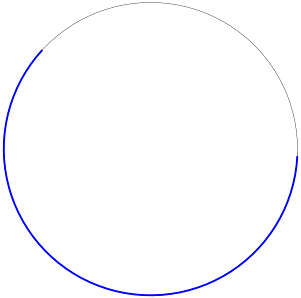
For the Fibonacci Hamiltonian $A = S_{-1} + M_v + S_1$, one gets

$$\sigma^{\text{op}}(A) = \left\{ S_{-1} + M_{v^\theta} + S_1,\ S_{-1} + M_{w^\theta} + S_1 \ :\ \theta \in [0,1) \right\},$$

where

$$v_n^\theta \ := \ \chi_{[1-\alpha,1)}(\theta + n\alpha \text{ mod } 1), \qquad w_n^\theta \ := \ \chi_{(1-\alpha,1]}(\theta + n\alpha \text{ mod } 1), \qquad n \in \mathbb{Z}.$$

In particular, $A \in \sigma^{\text{op}}(A)$; so A is self-similar.



What do we want from the Fibonacci Hamiltonian?

A lot is known of the spectrum of A; it is

- a Cantor set on the real line
- of Lebesgue measure zero,
- there is no point spectrum (w.r.t. ℓ^2)
- in fact, the spectrum is purely singular continuous...

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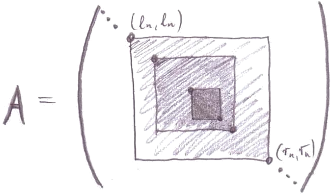
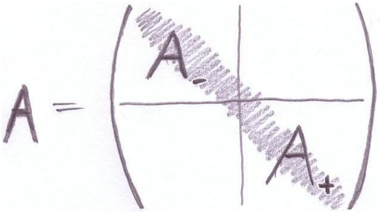
Our focus: **Applicability of the FSM** with arbitrary cut-off points.





We show this via **invertibility** of B, B_+ and B_- for all $B \in \sigma^{\mathrm{op}}(A)$, including $B = A$ (i.e. 0 is not in the spectrum of any of these operators).

Sketch of proof

To show that A is invertible on ℓ^2 (hence on any ℓ^p), we show that

- A is Fredholm ($\Rightarrow$ closed range)
- A is injective on ℓ^2
- A^* is injective on ℓ^2

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We demonstrate this for $B = A$ (so that $B_+ = A_+$):

Let $A_+x = o$, i.e.

$$\begin{pmatrix} 1 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & 1 & 1 & & & \\ & & 1 & 1 & 1 & & \\ & & & 1 & 0 & 1 & \\ & & & & 1 & 1 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

In short:

$$x_{n-1} + v_n x_n + x_{n+1} = 0 \quad \text{for all } n \in \mathbb{N}.$$

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n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
v_n	1	0	1	1	0	1	0	1	1	0	1	1	0	1	0	1	1	0	1	0
x_n	1	-1	-1	2	-1	-2	3	2	-5	3	5	-8	3	8	-11	-8	19	-11	-19	30

In short:

$$x_{n-1} + v_n x_n + x_{n+1} = 0 \qquad \text{for all } n \in \mathbb{N}.$$

Starting with $x_0 = 0$ and $x_1 = 1$ (w.l.o.g.) this is a 2-term recurrence for $x_2, x_3, \ldots$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
v_n	1	0	1	1	0	1	0	1	1	0	1	1	0	1	0	1	1	0	1	0
x_n	1	−1	−1	2	−1	−2	3	2	−5	3	5	−8	3	8	−11	−8	19	−11	−19	30

$\cdots \quad \Longrightarrow \quad x \notin \ell^\infty \quad \Longrightarrow \quad A_+$ is injective on ℓ^∞

Sketch of proof

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v_n	1	0	1	1	0	1	0	1	1	0	1	1	0	1	0	1	1	0	1	0
x_n	1	-1	-1	2	-1	-2	3	2	-5	3	5	-8	3	8	-11	-8	19	-11	-19	30

$\cdots \quad \Longrightarrow \quad x \notin \ell^\infty \quad \Longrightarrow \quad A_+$ is injective on ℓ^∞

Similarly: B_+ and B_- are injective on ℓ^∞ for all $B \in \sigma^{\mathrm{op}}(A)$.

$\Longrightarrow \quad B_+$ and B_- are invertible for all $B \in \sigma^{\mathrm{op}}(A)$.

In short:

$$x_{n-1} + {\color{red}v_n} x_n + x_{n+1} = 0 \qquad \text{for all } n \in \mathbb{N}.$$

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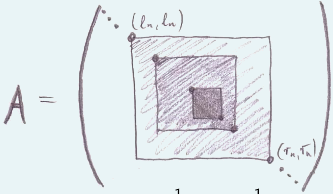
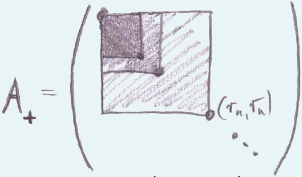
Theorem (ML, Söding 2016)

The FSM is applicable, with arbirtrary cut-off points, to A and also to A_+.



$A_n^{-1} \to A^{-1}$



$A_{+,n}^{-1} \to A_+^{-1}$

Thank you!

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