On the approximation theory of C_0-semigroups

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IM PAN

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Set-up: three classical approximations

Theorem Let $(e^{-tA})_{t\geq 0}$ be a bounded C_0-semigroup on a Banach space X. Then for every $x \in X$,

a) [Yosida's approximation]

$$e^{-tA}x = \lim_{n\to\infty} e^{-ntA(n+A)^{-1}}x;$$

b) [Dunford-Segal's approximation]

$$e^{-tA}x = \lim_{n\to\infty} e^{-nt(1-e^{-A/n})}x$$

c) [Euler's approximation]

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Problem 1. Is it possible to unify a), b) and c) ?

Some motivations to study semigroup approximations

- indispensable in semigroup theory: Hille-Yosida theorem, Trotter-Kato theorems, Chernoff theorems ...
- of importance in numerical analysis of PDE's: Brenner, Hersh, Kato, Thomee, Wahlbin, ...
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It was treated mainly for analytic semigroups: Crouzeix, Fujita, Larsson, le Roix, C. Palencia, Thomée, Wahlbin, Zagrebnov and many others ...

Mainly scattered facts ...

An example of history: Euler's approximation

Theorem Let $-A$ be the generator of a bounded C_0-semigroup $(e^{-tA})_{t\geq 0}$ on a Banach space X. Then there exists $c > 0$ such that for all $n \in \mathbb{N}$ and $t > 0$,

(i) (Brenner-Thomée)

$$\|e^{-tA}x - (1 + tA/n)^{-n}x\| \leq c\left(\frac{t}{\sqrt{n}}\right)^2 \|A^2x\|, \qquad x \in \mathrm{dom}\,(A^2);$$

(ii) (Flory-Weis)

$$\|e^{-tA}x - (1 + tA/n)^{-n}x\| \leq c\frac{t}{\sqrt{n}} \|Ax\|, \qquad x \in \mathrm{dom}\,(A);$$

(iii) (Kovacs)

$$\|e^{-tA}x - (1 + tA/n)^{-n}x\| \leq c\left(\frac{t}{\sqrt{n}}\right)^\alpha \|x\|_{\alpha,2,\infty}, \qquad x \in X_{\alpha,2,\infty},$$

where $X_{\alpha,2,\infty}, 0 < \alpha \leq 2$, is a certain Favard space.

Observation for the scalar exponent

Consider

$$\Delta_{t,n}^{\varphi}(z) := e^{-nt\varphi(z/n)} - e^{-tz} \to 0, \qquad n \to \infty,$$

or

$$E_{t,n}^{\varphi}(z) := e^{-n\varphi(tz/n)} - e^{-tz} \to 0, \qquad n \to \infty,$$

for all $t > 0$ and $z > 0$, with φ being an appropriate function.

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This way we get all three approximations:

- Yosida approximation $\longrightarrow \Delta_{t,n}^{\varphi}$ with $\varphi(z) = z/(z+1)$
- Dunford-Segal approximation $\longrightarrow \Delta_{t,n}^{\varphi}$ with $\varphi(z) = 1 - e^{-z}$
- Euler approximation $\longrightarrow E_{t,n}^{\varphi}$ with $\varphi(z) = \log(1+z)$

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Note that all of φ's are Bersntein and try to plug in A instead of z

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Start from the generator $-A$ of a bounded C_0-semigroup on a Banach space and consider

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- estimate $\Delta_t^{\varphi}(A)$
- do scaling: $\Delta_{nt}^{\varphi}(z/n) = \Delta_{t,n}^{\varphi}, \quad \Delta_n^{\varphi}(tz/n) = E_{t,n}^{\varphi}(z).$
- obtain universal approximation formulas with (optimal) convergence rates

Bernstein functions: why ?

A function $\varphi : (0, \infty) \mapsto (0, \infty)$ is *completely monotone* if there exists a positive measure μ such that

$$\varphi(z) = \int_0^\infty e^{-zt}\, d\mu(t), \qquad z > 0.$$

A function φ is *Bernstein* if φ' is completely monotone.

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A family of positive Borel measures $(\mu_t)_{t\geq 0}$ on $\mathbb{R}_+$ is called a vaguely continuous convolution semigroup of subprobability measures if for all $s, t \geq 0$:

$$\mu_t(\mathbb{R}_+) \leq 1, \qquad \mu_{t+s} = \mu_t * \mu_s, \qquad \text{and} \qquad \mathrm{wek}* - \lim_{t\to 0+} \mu_t = \delta_0.$$

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Theorem(Bochner subordination) A function φ is Bernstein if and only if there exists a vaguely continuous semigroup $(\mu_t)_{t\geq 0}$ of subprobability measures on $\mathbb{R}_+$ such that for all $t \geq 0$:

$$e^{-t\varphi(z)} = \int_0^\infty e^{-sz}\, d\mu_t(s), \qquad z \in \mathbb{C}_+.$$

Bernstein functions: remarks and class Φ

1. A function $f \in C^\infty(0,\infty)$ is completely monotone if

$$f(t) \geq 0 \quad \text{and} \quad (-1)^n \frac{d^n f(t)}{dt^n} \geq 0 \qquad \text{for all } n \in \mathbb{N} \text{ and } t > 0.$$

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2. **A flavour of positivity:** If f is compl. monotone and φ is Bernstein $\Rightarrow f \circ \varphi$ is compl. monotone. If, in addition, f is bdd $\Rightarrow f \circ \varphi$ is bdd and compl. monotone, and

$$\|f \circ \varphi\|_{\mathrm{A}^1_+(\mathbb{C}_+)} = (f \circ \varphi)(0+) \leq \|f\|_{\mathrm{A}^1_+(\mathbb{C}_+)}.$$

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3. **Definition of our function class**

$$\Phi := \{\varphi \text{ is Bernstein} : \ \varphi(0) = 0, \quad \varphi'(0+) = 1, \quad |\varphi''(0+)| < \infty\}.$$

Hille-Phillips functional calculus

Let $\mathrm{M}(\mathbb{R}_+)$ be the Banach algebra of bounded Radon measures on $\mathbb{R}_+$.
Laplace transform of $\mu \in \mathrm{M}(\mathbb{R}_+)$:

$$(\mathcal{L}\mu)(z) := \int_0^\infty e^{-sz}\,\mu(ds), \qquad z \in \mathbb{C}_+.$$

$$\begin{aligned} \mathrm{A}^1_+(\mathbb{C}_+) &:= \{\mathcal{L}\mu : \mu \in \mathrm{M}(\mathbb{R}_+)\} \\ \|\mathcal{L}\mu\|_{\mathrm{A}^1_+(\mathbb{C}_+)} &:= \|\mu\|_{\mathrm{M}(\mathbb{R}_+)} = |\mu|(\mathbb{R}_+). \end{aligned}$$

$(\mathrm{A}^1_+(\mathbb{C}_+), \|\cdot\|_{\mathrm{A}^1_+(\mathbb{C}_+)})$ is a commutative Banach algebra with pointwise multiplication, the Laplace transform

$$\mathcal{L} : \mathrm{M}(\mathbb{R}_+) \mapsto \mathrm{A}^1_+(\mathbb{C}_+)$$

is an isometric isomorphism.

Hille-Phillips calculus:cd

Let $-A$ be the generator of a bounded C_0-semigroup $(e^{-tA})_{t\geq 0}$ on a Banach space X.
The mapping

$$g = \mathcal{L}\mu \quad \mapsto \quad g(A) := \int_{\mathbb{R}_+} e^{-tA}\,\mu(\mathrm{d}t)$$

defines a continuous algebra homomorphism ("functional calculus") from $\mathrm{A}^1_+(\mathbb{C}_+)$ into the Banach space of bounded linear operators on X satisfying

$$\|g(A)\| \leq (\sup_{t\geq 0} \left\|e^{-tA}\right\|)\, \|g\|_{\mathrm{A}^1_+(\mathbb{C}_+)}, \qquad g \in \mathrm{A}^1_+(\mathbb{C}_+).$$

This homomorphism is called the *Hille–Phillips* (HP) functional calculus for A.

Rates for approximations: general case

The Hille-Phillips calculus (and Bochner subord.) yields

Theorem Let $\varphi \in \Phi$, $\alpha \in (0,2]$ and $M := \sup_{t\geq 0} \|e^{-tA}\|$. Then for all $x \in \mathrm{dom}(A^\alpha)$, $t > 0$, and $n \in \mathbb{N}$,

$$\|e^{-nt\varphi(A/n)}x - e^{-tA}x\| \leq 8M \left(\frac{t|\varphi''(0+)|}{n}\right)^{\alpha/2} \|A^\alpha x\|,$$

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The main point:

$$\frac{e^{-\varphi(z)} - e^{-z}}{z^2} \quad \text{is a bdd completely monotone function}$$

Remark

- Theorem $\Rightarrow$ the convergence of $\Delta^{\varphi}_{t,n}(A)$ and $\Delta^{\varphi}_{t,n}(A)$ in SOT.
- The domain $(0,2]$ for α in Theorem cannot, in general, be enlarged.

A corollary: rates for three classical approximations

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a) [Yosida's approximation]

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Can Theorem be improved for analytic semigroups ? Rich history ...

Rates for approximations: (sectorially) bounded analytic semigroups

Theorem Let $\varphi \in \Phi$ and $\alpha \in [0,1]$. Then there exists $C > 0$ such that

$$\|(e^{-nt\varphi(A/n)} - e^{-tA})x\| \le \frac{C}{nt^{1-\alpha}} \|A^\alpha x\|,$$

and

$$\|(e^{-n\varphi(tA/n)} - e^{-tA})x\| \le \frac{Ct^\alpha}{n} \|A^\alpha x\|,$$

for all $x \in \mathrm{dom}(A^\alpha)$, $n \in \mathbb{N}$ and $t > 0$.

Note: α can be zero = convergence with (optimal) rates in the uniform op. topology !

A corollary for three classical approximations

Corollary Let $\alpha \in [0, 1]$. Then there exists $C > 0$ such that

a) [Yosida's approximation]

$$\|e^{-tA}x - e^{-ntA(n+A)^{-1}}x\| \leq C(nt^{1-\alpha})^{-1}\|A^\alpha x\|;$$

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c) [Euler's approximation]

$$\|e^{-tA}x - (1 + t/nA)^{-n}x\| \leq C(nt^{-\alpha})^{-1}\|A^\alpha x\|,$$

for all $t > 0$, $n \in \mathbb{N}$ and $x \in \mathrm{dom}(A^\alpha)$.

Optimality

Theorem Let $\overline{\mathrm{ran}}\,(A) = X$, and let $\varphi \in \Phi, \varphi(z) \not\equiv z$.

(i) If $\{|s| : s \in \mathbb{R}, \, is \in \sigma(A)\} = \mathbb{R}_+$, then there exist $c > 0$ and $T > 0$ such that for every $\alpha \in (0, 2]$ and all $t \geq T$,

$$\begin{aligned} \|A^{-\alpha}(e^{-nt\varphi(A/n)} - e^{-tA})\| &\geq c\left(\frac{t}{n}\right)^{\alpha/2}, \\ \|A^{-\alpha}(e^{-n\varphi(tA/n)} - e^{-tA})\| &\geq c\left(\frac{t^2}{n}\right)^{\alpha/2}. \end{aligned}$$

(ii) If $\mathbb{R}_+ \subset \sigma(A)$, then there exist $T > 0$ and $c > 0$ such that for every $\alpha \in [0, 1]$ and all $t \geq T$,

$$\begin{aligned} \|A^{-\alpha}(e^{-nt\varphi(A/n)}x - e^{-tA})\| &\geq cn^{-1}t^{\alpha-1}, \\ \|A^{-\alpha}(e^{-n\varphi(tA/n)} - e^{-tA})\| &\geq cn^{-1}t^{\alpha}. \end{aligned}$$

A yet more general setting: motivation

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3. For every bounded CM function g with $g'(0+) = -1$:

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(Chernoff type approximation)

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4. Not every bounded CM function g is as in 1:

$$\text{e.g.} \qquad \text{if} \qquad g(z) = \int_0^T e^{-sz} d\nu(s), \qquad z > 0, \qquad T < \infty.$$

Some known approximations outside of the scope:

$$g(z) = \frac{1 - e^{-2z}}{2z}, \qquad g_t(z) = 1 - t + t\exp(-z/t), t \in (0,1), \ldots$$

Approximation of general C_0-semigroups: first order formulas

Theorem[Gomilko-Kosowicz-T.] Let $-A$ be the generator of a bounded C_0-semigroup $(e^{-tA})_{t\geq 0}$ on X, and let $(g_t)_{t\geq 0}$ be a family of CM functions such that

$$g_t(0+) = 1, \qquad g_t'(0+) = -1 \quad \text{and} \quad g_t''(0+) < \infty.$$

for every $t \geq 0$. Let $M := \sup_{t\geq 0} \|e^{-tA}\|$.

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Then for all $n \in \mathbb{N}$ and $t \geq 0$,

$$\|g_t^n(tA/n)x - e^{-tA}x\| \leq M\frac{(g_t''(0) - 1)}{2}\frac{t^2}{n}\|A^2x\|, \quad x \in \mathrm{dom}(A^2),$$

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and, if $\alpha \in (0, 2)$,

$$\|g_t^n(tA/n)x - e^{-tA}x\| \leq 4M \left((g_t''(0) - 1) \frac{t^2}{n} \right)^{\frac{\alpha}{2}} \|A^\alpha x\|, \quad x \in \mathrm{dom}(A^\alpha).$$

Approximation of analytic C_0-semigroups

Theorem Let $(g_t)_{t\geq 0}$ be as before, and let $-A$ be the generator of a sect. bdd analytic C_0-semigroup $(e^{-tA})_{t\geq 0}$ on X. Define

$$M_\beta := \sup_{t\geq 0} \|t^\beta A^\beta e^{-tA}\|, \qquad \beta = 0, 1, 2.$$

Then for all $n \in \mathbb{N}$ and $t > 0$,

$$\|g_t^n(tA/n)x - e^{-tA}x\| \leq (2M_0 + 3M_1/2)\frac{(g_t''(0) - 1)}{n} t\|Ax\|, \quad x \in \mathrm{dom}(A),$$

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Moreover, for all $n \in \mathbb{N}$ and $t > 0$,

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$$\|g_t^n(tA/n)x - e^{-tA}x\| \leq (2M_0 + 3M_1/2)\frac{(g_t''(0) - 1)}{n} t\|Ax\|, \quad x \in \mathrm{dom}(A),$$

Moreover, for all $n \in \mathbb{N}$ and $t > 0$,

$$\|g_t^n(tA/n) - e^{-tA}\| \leq K\frac{(g_t''(0) - 1)}{n},$$

and, if $\alpha \in [0, 1]$,

$$\|g_t^n(tA/n)x - e^{-tA}x\| \leq 4K\frac{(g_t''(0) - 1)}{n} t^\alpha \|A^\alpha x\|, \quad x \in \mathrm{dom}(A^\alpha),$$

where $K = 3M_0 + 3M_1 + M_2/2$.

Approximation of general C_0-semigroups: higher-order formulas

Theorem Let $(g_t)_{t\geq 0}$ be family of CM functions such that

$$g_t(0+) = 1, \qquad g_t'(0+) = -1 \qquad |g_t^{(k)}(0+)| < \infty, \qquad k = 3, 4,$$

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for every $t \geq 0$. Let $M := \sup_{t\geq 0} \|e^{-tA}\|$. Then for all $t > 0$ and $n \in \mathbb{N}$,

$$\|g_t^n(tA/n) - (2n)^{-1}(g_t''(0) - 1)t^2 e^{-tA} A^2 x\| \\ \leq MC(g_t)t^3 n^{-3/2}\|A^3 x\|, \quad x \in \mathrm{dom}(A^3),$$

where $C(g_t)$ is a finite linear comb. of the derivatives of g_t at 0.
Moreover, for all $t > 0$ and $n \in N$,

$$\|g_t^n(tA/n)x - (2n)^{-1}(g_t''(0) - 1)t^2 e^{-tA} A^2 x\| \\ \leq MC_1(g_t)n^{-2} t^3(\|A^3 x\|, +t\|A^4 x\|), \quad x \in \mathrm{dom}(A^4),$$

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The “best” (final ?) Euler’s formula

Theorem Let $-A$ be the generator of a sect. bdd holomorphic C_0-semigroup $(e^{-tA})_{t\geq 0}$ on X. Then for all $\alpha \in [0,1]$, $t > 0$, and $n \in \mathbb{N}$,

$$\left\| e^{-tA}x - \left(1 + \frac{t}{n}A\right)^{-n} x \right\| \leq M_{2-\alpha} r_n t^\alpha \|A^\alpha x\|, \quad x \in \mathrm{dom}(A^\alpha),$$

where

$$r_n \leq \frac{1}{n}, \quad r_n = \frac{1}{2n} + \frac{r_{0,n}}{n^2} \quad |r_{0,n}| \leq C.$$

Moreover, there exists $C > 0$ such that for all $\alpha \in [0,1]$, $t > 0$, and $n \in \mathbb{N}$,

$$\left\| e^{-tA}x - \left(1 + \frac{t}{n}A\right)^{-n} x - \frac{t^2 A^2 x}{2n} e^{-tA} x \right\|$$
$$\leq C[M_{3-\alpha} + M_{4-\alpha}] \frac{t^\alpha}{n^2} \|A^\alpha x\|, \quad x \in \mathrm{dom}(A^\alpha).$$

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d) we get best (?) constants and optimal higher order approximations