An asymptotic lower bound for the norm of the Laplace operator on a space of polynomials

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Introduction

A. Böttcher, C.Rebs:
On the constants in Markov inequalities for the Laplace operator on polynomials with the Laguerre norm,
Asymptotic Analysis 101 (2017), 227-239.

For $n, N \in \mathbb{N}$ we define $\mathcal{P}_n^N$ as the finite dimensional linear space of all complex polynomials f of the form

$$f(t_1, \ldots, t_N) = \sum_{(\frac{i_1}{n}, \ldots, \frac{i_N}{n}) \in [0,1]^N} f_{i_1, \ldots, i_N} t_1^{i_1} \ldots t_N^{i_N},$$

$f_{i_1, \ldots, i_N} \in \mathbb{C}$.
We equip these space with the Laguerre norm

$$\|f\|^2 := \int_{(0,\infty)^N} |f(t_1, \ldots, t_N)|^2 e^{-t_1} \ldots e^{-t_N} \, dt_1 \ldots dt_N.$$

The Laplace operator

$$\Delta = \frac{\partial^2}{\partial t_1^2} + \ldots + \frac{\partial^2}{\partial t_N^2} : \mathcal{P}_n^N \to \mathcal{P}_n^N$$

is a bounded linear operator on $\mathcal{P}_n^N$.
There exists a constant $C = C(n, N)$, such that

$$\|\Delta f\| \leq C\|f\|$$

holds for all $f \in \mathcal{P}_n^N$.
The best constant with this property is $C = \|\Delta\|$.
Aim: Calculate (asymptotic) bounds for $\|\Delta\|$

Main result:

Theorem 1. *Let* $n, N \in \mathbb{N}$ *and let* ω_0 *be the positive solution of the equation* $2 + 2\cosh(\omega) - \omega\sinh(\omega) = 0$*. Then we have for the operator norm of the Laplace operators on* $\mathcal{P}_n^N$ *with respect to the Laguerre norm*

$$\|\Delta\| \geq \frac{N}{\omega_0^2}(n+1)^2 + o\Big((n+1)^2\Big)$$

for $n \to \infty$*.*

With $\frac{2}{\omega_0^2} = 0,34741\ldots$ this implies

$$\liminf_{n\to\infty} \frac{\|\Delta\|}{Nn^2} \geq 0.1737.$$

The matrix representation of Δ

The norm $\|\Delta\|$ is the largest singular value of the matrix representation $[\Delta]$ of Δ in an orthonormal basis of $\mathcal{P}_n^N$.
We set $\mathcal{P}_n := \mathcal{P}_n^1$. For $k \in \mathbb{N}_0$ we set

$$L_k(t) := 1 - \binom{k}{1}\frac{t}{1!} + \binom{k}{2}\frac{t^2}{2!} - \cdots + (-1)^k\binom{k}{k}\frac{t^k}{k!}$$

and get an orthonormal basis $\{L_0, L_1, \ldots, L_n\}$ is $\mathcal{P}_n$.

The ordinary differential operator D^2 is defined by
$D^2 : \mathcal{P}_n \to \mathcal{P}_n, f \mapsto f''$.

[1] L. F. Shampine: *Some L_2 Markoff inequalities*,
J. Res. Nat. Bur. Standards 69B (1965), 155-158:

The matrix representation of D^2 is the $(n+1) \times (n+1)$ matrix

$$\left[D^2\right] = \begin{pmatrix} 0 & 0 & 1 & 2 & \dots & n-2 & n-1 \\ & 0 & 0 & 1 & \dots & n-3 & n-2 \\ & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & 0 & 1 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}.$$

We identify $\mathcal{P}_n^N$ with $\bigotimes_{j=1}^N \mathcal{P}_n$. The polynomials $L_{j_1} \otimes \cdots \otimes L_{j_N}$ with $j_1, \ldots, j_N \in \{0, \ldots, n\}$ form an orthonormal basis in $\mathcal{P}_n^N$. The matrix representation of the Laplace operator in this basis is

$$
\begin{aligned}
[\Delta] &= \left[D^2 \otimes I \otimes \ldots \otimes I + I \otimes D^2 \otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes D^2\right] \\
&= \left[D^2\right] \otimes I_{n+1} \otimes \ldots \otimes I_{n+1} + I_{n+1} \otimes \left[D^2\right] \otimes I_{n+1} \otimes \ldots \otimes I_{n+1} \\
&\quad + \ldots + I_{n+1} \otimes \ldots \otimes I_{n+1} \otimes \left[D^2\right] \\
&= \sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes \left[D^2\right] \otimes I_{(n+1)^{N-k}}.
\end{aligned}
$$

Proof of the main result

For a $n \times n$ matrix A we set $H(A) := \frac{1}{2}(A + A^*)$.

[2] R. A. Horn, C. R. Johnson: *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, Sao Paulo, 8th printing, 2007:

Theorem 2: [Corollary 3.1.5 in [2]]
For a $n \times n$ matrix A we denote by $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_n(A)$ the singular values A and by $\lambda_1(H(A)) \geq \ldots \geq \lambda_n(H(A))$ the eigenvalues of $H(A)$. Then we have the estimate

$$\sigma_k(A) \geq \lambda_k(H(A))$$

for all $k = 1, \ldots, n$.

For a $n \times n$ matrix A and a $m \times m$ matrix B we define the Kronecker sum

$$A \oplus B := (I_m \otimes A) + (B \otimes I_n)\,.$$

Theorem 3: [Theorem 4.4.5 in [2]]
If λ is an eigenvalue of A and if μ is an eigenvalue of B, then $\lambda + \mu$ is an eigenvalue of $A \oplus B$.
If ν is an eigenvalue of $A \oplus B$, then there exist eigenvalues λ of A and μ of B such that $\nu = \lambda + \mu$.

We need Kronecker sums with more then two summands:
For matrices A_1 of order n_1, ..., A_N of order n_N we set

$$A_1 \oplus \ldots \oplus A_N := A_1 \oplus (A_2 \oplus (A_3 \oplus (\ldots \oplus (A_{N-1} \oplus A_N) \ldots).$$

For the Kronecker sum with N summands we get the formula

$$A_1 \oplus \ldots \oplus A_N = \sum_{j=1}^{N} I_{\prod_{k=j+1}^{N} n_k} \otimes A_j \otimes I_{\prod_{k=1}^{j-1} n_k},$$

where we set $\prod_{k=j}^{l} n_k = 1$ for $l < j$.

Eigenvalues:
For eigenvalues λ_1 of A_1, λ_2 of A_2, ..., λ_N of A_N, $\lambda_1 + \ldots + \lambda_N$ is an eigenvalue of $A_1 \oplus \ldots \oplus A_N$.
Every eigenvalue λ of $A_1 \oplus \ldots \oplus A_N$ is a sum $\lambda = \lambda_1 + \ldots + \lambda_N$ of eigenvalues λ_k of A_k, $k = 1, \ldots, N$.

Now we get an estimate for the norm of the Laplace operator:

$$\begin{aligned}
\|\Delta\| &= \sigma_{\text{max}}([\Delta]) \geq \lambda_{\text{max}}\left(\frac{1}{2}\left([\Delta]+[\Delta]^T\right)\right) \\
&= \frac{1}{2}\lambda_{\text{max}}\left(\sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes \left[D^2\right] \otimes I_{(n+1)^{N-k}}\right. \\
&\quad \left. + \sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes \left[D^2\right]^T \otimes I_{(n+1)^{N-k}}\right) \\
&= \frac{1}{2}\lambda_{\text{max}}\left(\sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes \left(\left[D^2\right]+\left[D^2\right]^T\right) \otimes I_{(n+1)^{N-k}}\right) \\
&= \frac{1}{2}\lambda_{\text{max}}\left(\left(\left[D^2\right]+\left[D^2\right]^T\right) \oplus \ldots \oplus \left(\left[D^2\right]+\left[D^2\right]^T\right)\right) \\
&= \frac{N}{2}\lambda_{\text{max}}\left(\left[D^2\right]+\left[D^2\right]^T\right).
\end{aligned}$$

We have to calculate the largest eigenvalue of

$$\left[D^2\right] + \left[D^2\right]^T = \begin{pmatrix} 0 & 0 & 1 & 2 & & \dots & n-2 & n-1 \\ 0 & 0 & 0 & 1 & & \dots & & n-2 \\ 1 & 0 & & & & & \vdots & \vdots \\ 2 & 1 & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & 1 & 2 \\ \vdots & \vdots & & & & & 0 & 1 \\ n-2 & & \dots & & 1 & 0 & 0 & 0 \\ n-1 & n-2 & \dots & & 2 & 1 & 0 & 0 \end{pmatrix}.$$

[3] J. M. Bogoya, A. Böttcher, S. M. Grudsky: *Eigenvalues of Hermitian Toeplitz matrices with polynomially increasing entries*, Journal of Spectral Theory 2 (2012), 267-292:

For $\alpha > 0$ we define the integral operator K_α on $L^2(0,1)$ as

$$(K_\alpha f)(x) := \int_0^1 |x-y|^\alpha f(y)\; dy$$

for $x \in (0,1)$. We denote by $\mu_1(K_\alpha) \geq \mu_2(K_\alpha) \geq \ldots > 0$ the positive eigenvalues of K_α and by $\mathcal{L}_+$ the index set of this eigenvalues.

Theorem 4: [Theorem 1.2 in [3]]
The operator K_1 has only one positive eigenvalue

$$\mu_1(K_1) = \frac{2}{\omega_0^2} = 0{,}34741\ldots.$$

Here is ω_0 the positive solution of the equation
$2 + 2\cosh(\omega) - \omega \sinh(\omega) = 0.$

We denote by $T_n[a_0, \ldots, a_{n-1}]$ the hermitian Toeplitz matrix

$$T_n = T_n[a_0, \ldots a_{n-1}] = \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ \bar{a}_1 & a_0 & \ldots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n-1} & \bar{a}_{n-2} & \ldots & a_0 \end{pmatrix}.$$

For the eigenvalues $\widetilde{\lambda}_1(T_n) \leq \widetilde{\lambda}_2(T_n) \leq \ldots \leq \widetilde{\lambda}_n(T_n)$ of T_n we have:

Theorem 5: [Theorem 1.1 in [3]]
We assume $a_k = k^\alpha + o(k^\alpha)$ for $k \to \infty$, $\alpha > 0$. Then the eigenvalues of $T_n = T_n[a_0, \ldots, a_{n-1}]$ satisfy, as $n \to \infty$,

$$\widetilde{\lambda}_{n+1-l}(T_n) = \mu_l(K_\alpha) n^{\alpha+1} + o(n^{\alpha+1})$$

for $l \in \mathcal{L}_+$.

We have $\left[D^2\right] + \left[D^2\right]^T = T_{n+1}[0, 0, 1, \ldots, n-1] = T_{n+1}[a_0, \ldots, a_n]$ with $a_k = k + o(k)$ for $k \to \infty$.
Now we apply Theorem 1.1 from [3] and get for the largest eigenvalue of $\left[D^2\right] + \left[D^2\right]^T$

$$\begin{aligned}
\lambda_{\max}\left(\left[D^2\right] + \left[D^2\right]^T\right) &= \widetilde{\lambda}_{n+1}\left(T_{n+1}[0, 0, 1, \ldots, n-1]\right) \\
&= \widetilde{\lambda}_{n+2-1}\left(T_{n+1}[0, 0, 1, \ldots, n-1]\right) \\
&= \mu_1(K_1)(n+1)^2 + o\left((n+1)^2\right) \\
&= \frac{2}{\omega_0^2}(n+1)^2 + o\left((n+1)^2\right). \qquad (n \to \infty)
\end{aligned}$$

This leads for all $N \in \mathbb{N}$ to the estimate

$$\|\Delta\| \geq \frac{N}{2}\lambda_{\max}\left(\left[D^2\right] + \left[D^2\right]^T\right) = \frac{N}{\omega_0^2}(n+1)^2 + o\left((n+1)^2\right)$$

for $n \to \infty$.

An upper bound

We have $\|\Delta\| = \left\|\sum_{k=1}^{N} I_{(n+1)^{k-1}} \otimes \left[D^2\right] \otimes I_{(n+1)^{N-k}}\right\| \leq N \left\|\left[D^2\right]\right\|$.
From

[1] L. F. Shampine: *Some L_2 Markoff inequalities*,
J. Res. Nat. Bur. Standards 69B (1965), 155-158:

we know

$$\frac{\left\|D^2\right\|}{n^2} \to \frac{1}{\mu^2} = 0.28441\ldots,$$

where μ is the smallest positive solution of $1 + \cos\mu\cosh\mu = 0$.
This leads to

$$0.1737 \leq \liminf_{n\to\infty} \frac{\|\Delta\|}{Nn^2} \leq \limsup_{n\to\infty} \frac{\|\Delta\|}{Nn^2} \leq 0.2845.$$

$N = 1$	n	10	50	100	500	1000
	$\frac{\|\Delta\|}{n^2}$	0.2829	0.2844	0.2844	0.2844	0.2844

$N = 2$	n	10	50	100	500	1000
	$\frac{\|\Delta\|}{2n^2}$	0.2107	0.2175	0.2183	0.2189	0.2190

$N = 3$	n	10	20	30	50	100
	$\frac{\|\Delta\|}{3n^2}$	0.1950	0.1993	0.2008	0.2020	0.2029

$N = 4$	n	10	20	30	50	100
	$\frac{\|\Delta\|}{4n^2}$	0.1855	0.1906	0.1923	0.1937	0.1947

$N = 5$	n	10	20	30	50
	$\frac{\|\Delta\|}{5n^2}$	0.1804	0.1857	0.1875	0.1889

$N = 6$	n	10	20	30
	$\frac{\|\Delta\|}{6n^2}$	0.1770	0.1824	0.1843

$N = 7$	n	10
	$\frac{\|\Delta\|}{7n^2}$	0.1745