Holomorphic functions which preserve holomorphic semigroups

Charles Batty (University of Oxford)

joint work with Alexander Gomilko (Torun) and Yuri Tomilov (IMPAN, Warsaw)

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Holomorphic semigroups

Bounded holomorphic C_0-semigroup on X:

$$T : \Sigma_\theta := \{z \in \mathbb{C} : |\arg z| < \theta\} \to \mathcal{B}(X), \quad \text{holomorphic}$$
$$\sup\{\|T(z)\| : z \in \Sigma_\theta\} < \infty, \quad T(z_1 + z_2) = T(z_1)T(z_2),$$
$$\lim_{z \to 0} \|T(z)x - x\| = 0.$$

Sectorial operator $A : D(A) \subset X \to X$,

$$\sigma(A) \subset \overline{\Sigma}_\theta, \qquad \|(\lambda + A)^{-1}\| \leq \frac{C_\theta}{|\lambda|} \quad \lambda \in \Sigma_{\pi-\theta}, \qquad 0 < \theta < \pi.$$

Sectorial angle of A: the infimum ω_A of all such $\theta \in (0, \pi)$

We assume (for convenience) that A has dense domain $D(A)$, and dense range. Then A is injective, and $A^{-1} : \mathrm{Ran}(A) \to X$ is sectorial of the same angle.

A is sectorial with $\omega_A < \pi/2$ if and only if $-A$ generates a bounded holomorphic semigroup (of angle $\pi/2 - \omega_A$).

Functional calculus

Let $\theta > \omega_A$. For many holomorphic $f : \Sigma_\theta \to \mathbb{C}$, one can define $f(A)$ as a closed operator. There are several different methods, but they are all consistent, and have reasonably good functional calculus properties

- Fractional powers A^α (Balakrishnan)
- Complete Bernstein functions (Hirsch)
- Bernstein functions (Bochner, Phillips, Schilling et al)
- Holomorphic functions with at most polynomial growth as $|z| \to \infty$ and $|z| \to 0$ (McIntosh, Haase)

If $f(A) \in \mathcal{B}(X)$ for all $f \in H^\infty(\Sigma_\theta)$, then A has *bounded* H^∞-*calculus* on Σ_θ

Question

Given sectorial A and holomorphic f, when is $f(A)$ sectorial?

More specifically,

Q1. For which f is $f(A)$ sectorial (with $\omega_{f(A)} \leq \omega_A$) for all sectorial A?

Q2. For which A is $f(A)$ sectorial for all suitable f?

Q1 might be considered for the class of all Banach spaces X, or just for Hilbert spaces or some other class.

The set of functions f as in Q1 is closed under sums, positive scalar multiples, reciprocals and composition.

$\mathcal{NP}_+$-functions

For Q1, f should be

- holomorphic on $\mathbb{C}_+ = \Sigma_{\pi/2}$
- map $\mathbb{C}_+$ to $\mathbb{C}_+$
- map $(0,\infty)$ to $(0,\infty)$

Such a function is a *positive real* function (Cauer, Brune; Brown) or an $\mathcal{NP}_+$-function. Any $\mathcal{NP}_+$-function maps Σ_θ into Σ_θ for each $\theta \in (0,\pi/2)$.

$\mathcal{NP}_+$ is closed under sums, positive scalar multiples, reciprocals, compositions. It consists of the functions of the form

$$f(z) = \int_{-1}^{1} \frac{2z}{(1+z^2) + t(1-z^2)}\, d\mu(t)$$

for some finite positive Borel measure μ on $[-1,1]$. So estimates for the integrand which are uniform in t provide estimates for $|f(z)|$ subject to $f(1) = 1$.

Question 2

For which A is $f(A)$ sectorial for all $f \in \mathcal{NP}_+$?

Theorem

Let A be a sectorial operator on a Banach space X with dense range and $\omega_A < \pi/2$, and let $\theta \in (\omega_A, \pi/2)$. Consider the following statements.

(i) *A has bounded H^∞-calculus on Σ_θ.*

(ii) *For every $f \in \mathcal{NP}_+$, $f(A)$ is a sectorial operator of angle (at most) ω_A.*

(iii) *For every $f \in \mathcal{NP}_+$, $-f(A)$ is the generator of a bounded C_0-semigroup.*

(iv) *A has bounded H^∞-calculus on $\mathbb{C}_+$.*

Then

$$\text{(i)} \implies \text{(ii)} \implies \text{(iii)} \iff \text{(iv)}.$$

If X is a Hilbert space, all four properties are equivalent.

Q1: Limits at 0 and ∞

Let $f \in \mathcal{NP}_+$, and let

$$f(0+) = \lim_{t \to 0+} f(t), \qquad f(\infty) = \lim_{t \to \infty} f(t)$$

if these limits exist in $[0, \infty]$.

Proposition

Let $f \in \mathcal{NP}_+$ be a function such that $f(\infty)$ does not exist in $[0, \infty]$, and let X be a Banach space with a conditional basis. There exists a sectorial operator A on X, with angle 0, such that $-f(A)$ does not generate a C_0-semigroup.

So we restrict attention to $\mathcal{NP}_+$-functions for which $f(0+)$ and $f(\infty)$ exist.

Bernstein functions

A C^∞-function $f : (0,\infty) \to (0,\infty)$ is a Bernstein function if

$$(-1)^{n-1} f^{(n)}(t) \geq 0, \qquad (n \geq 1, t > 0).$$

Equivalently, there is a positive measure μ with

$$f(t) = a + bt + \int_0^\infty \left(1 - e^{-st}\right) \, d\mu(s), \quad \int_0^\infty \frac{s}{1+s} \, d\mu(s) < \infty.$$

If $-A$ generates a bounded C_0-semigroup T "$T(s) = e^{-sA}$",

$$f(A)x = ax + bAx + \int_0^\infty (x - T(s)x) \, d\mu(s), \qquad x \in D(A).$$

Q.1 for Bernstein functions

Question posed by Kishimoto and Robinson in 1981 (slightly vaguely)

Positive answers:

Balakrishnan (1960): fractional powers

Hirsch (1973): complete Bernstein functions (without preservation of angle)

Berg–Boyadzhiev–deLaubenfels (1993): preservation of angles ($< \pi/2$) for complete Bernstein functions, and partial results for some other Bernstein functions

Gomilko–Tomilov (2015): All Bernstein functions (with angle).

Now: 3 proofs

A resolvent formula for scalar functions

Let $f \in \mathcal{NP}_+$ and assume that $f(\infty)$ exists. Let $q > 2$, $z \in \Sigma_{\pi/q}$, $\lambda \in \Sigma_{\pi-\pi/q}$. Then

$$
\begin{aligned}
&(\lambda + f(z))^{-1} \\
&= \frac{1}{\lambda + f(\infty)} + \frac{q}{\pi} \int_0^\infty \frac{\operatorname{Im} f(te^{i\pi/q})\, t^{q-1}}{(\lambda + f(te^{i\pi/q}))(\lambda + f(te^{-i\pi/q}))(t^q + z^q)}\, dt,
\end{aligned}
$$

where the integral may be improper.

We would like to replace λ by a sectorial operator A, but does the integral converge in any sense? Can it be estimated in a way which shows that $f(A)$ is sectorial?

Resolvent formula for operators

For a sectorial operator A, we want the formula

$$
\begin{aligned}
&(\lambda + f(A))^{-1} \\
&= \frac{1}{\lambda + f(\infty)} + \frac{q}{\pi} \int_0^\infty \frac{\operatorname{Im} f(te^{i\pi/q})\, t^{q-1}}{(\lambda + f(te^{i\pi/q}))(\lambda + f(te^{-i\pi/q}))} (t^q + A^q)^{-1}\, dt.
\end{aligned}
$$

Theorem

Assume that $f \in \mathcal{NP}_+$, and

$$
\int_0^\infty \frac{|\operatorname{Im} f(te^{i\beta})|}{(r + f(t))^2} \frac{dt}{t} \le \frac{C_\beta}{r}, \qquad r > 0,\ \beta \in (0, \pi/2). \qquad (\mathcal{E})
$$

1. *$f(0+)$ and $f(\infty)$ exist.*
2. *If A is sectorial of angle $\omega_A < \pi/2$, then the resolvent formula above holds and $f(A)$ is sectorial of angle at most ω_A.*

Another condition

The condition $(\mathcal{E})$ on f is preserved by sums, positive scalar multiples, reciprocals, and $f \mapsto f(1/z)$.

$f \in \mathcal{NP}_+$ satisfies $(\mathcal{D})$ if, for each $\beta \in (0, \pi/2)$ there exist $a, b, c, a', b', c' > 0$ such that

- f is monotonic on $(0, a/b)$ and $|\operatorname{Im} f(te^{i\beta})| \le ct|f'(bt)|$ for $t < a/b$, and
- f is monotonic on $(a'/b', \infty)$ and $|\operatorname{Im} f(te^{i\beta})| \le c't|f'(b't)|$ for $t > a'/b'$.

Theorem

1. *Any Bernstein function satisfies $(\mathcal{D})$, with $a = b = a' = b' = 1$.*
2. *Assume that f satisfies $(\mathcal{D})$. Then f satisfies $(\mathcal{E})$. Hence $f(A)$ is sectorial whenever A is sectorial with $\omega_A < \pi/2$.*

Examples of $(\mathcal{D})$

z and $1 - e^{-z}$ are both Bernstein functions, and so are their square roots. Their geometric mean $\sqrt{z(1 - e^{-z})}$ is not Bernstein, but it is $\mathcal{NP}_+$ and it satisfies $(\mathcal{D})$.

In fact, if $f_1, \dots, f_n$ are Bernstein, and the product $f_1 \cdots f_n$ is $\mathcal{NP}_+$ then the product satisfies $(\mathcal{D})$.

In particular the geometric mean of any number of Bernstein functions satisfies $(\mathcal{D})$.

If f is Bernstein and $\alpha \in (0, 1)$, then

$$g_\alpha(z) := [f(z^\alpha)]^{1/\alpha}$$

is $\mathcal{NP}_+$ and satisfies $(\mathcal{D})$. If $\alpha \in (0, 1/2]$ then g_α is Bernstein, but this is not known for $\alpha \in (1/2, 1)$.

A formula of Boyadzhiev (2002)

Let $0 < \psi < \theta < \pi/2$, $g \in H^\infty(\Sigma_\theta)$, vanishing at infinity. Assume that

$$\|g'_\psi\|_1 := \frac{1}{2}\int_0^\infty \left| e^{i\psi} g'(te^{i\psi}) + e^{-i\psi} g'(te^{-i\psi}) \right| \, dt < \infty.$$

Let

$$V_\psi(z,t) = \frac{t}{2}\left(e^{-i\psi}(z - te^{-i\psi})^{-1} + e^{i\psi}(z - te^{i\psi})^{-1} \right)$$

Then

$$g(z) = \int_0^\infty V_\psi(z,t)(g'_\psi * k_\psi)(\log t)\,\frac{dt}{t}.$$

These formulas hold if z is replaced by a sectorial operator A of angle less than ψ. From this one can deduce that $g(A)$ is bounded and $\|g(A)\| \le C_{A,\psi}\|g'_\psi\|_1$.

Consequences

Third proof of the theorem of Gomilko and Tomilov: $g(z) = (\lambda + f(z))^{-1}$ where f is Bernstein

A proof of a theorem of Vitse (2005): $F(A)$ is bounded when $F \in B^{\infty}_{\infty 1}$, the analytic Besov space of functions $F \in H^{\infty}(\mathbb{C}_+)$ such that $\int_0^{\infty} \|F'(t + i\cdot)\|_{\infty}\, dt < \infty$