On the structure of Hausdorff moment sequences

B. Fritzsche B. Kirstein C. Mädler

Mathematisches Institut
Universität Leipzig

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Moment sequences

Question

Given a sequence

$$s_0, s_1, s_2, \ldots, s_m, \ldots$$

of real numbers, find a representing measure σ on $\Omega \subseteq \mathbb{R}$:

$$s_j = \int_\Omega x^j \sigma(\mathrm{d}x), \qquad j = 0, 1, 2, \ldots, m, \ldots$$

Classical moment problems

$\Omega = \mathbb{R}$	H. L. Hamburger	1889–1956
$\Omega = [0, \infty)$	T. J. Stieltjes	1856–1894
$\Omega = [0, 1]$	F. Hausdorff	1868–1942

Non-negative hermitian measures

- $\mathfrak{B}$: the σ-algebra of all Borel sets in $\Omega \subseteq \mathbb{R}$
- $\mathbb{C}^{p \times q}$: the set of all complex $p \times q$ matrices
- $A \succcurlyeq B$: A and B hermitian matrices with $A - B$ non-negative hermitian (*Löwner semi ordering*)
- $A \succcurlyeq \mathbb{O}$: matrix A is non-negative hermitian (= positive-semidefinite)

Definition

We call $\mu \colon \mathfrak{B} \to \mathbb{C}^{q \times q}$ a non-negative hermitian $q \times q$ measure on Ω if:

- $\mu(B) \succcurlyeq \mathbb{O}$ for all $B \in \mathfrak{B}$ and
- μ is σ-additive: $\mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$ for all sequences $(B_n)_{n=1}^{\infty}$ of pairwise disjoint sets from $\mathfrak{B}$.

Truncated matricial moment problem

Let $m \in \{0, 1, 2, \ldots\}$ and let $(s_j)_{j=0}^m$ be a sequence from $\mathbb{C}^{q \times q}$:

Moment problem $\mathrm{M}[\Omega, (s_j)_{j=0}^m]$

Describe the set $\mathcal{M}_{\succcurlyeq}^q[\Omega, (s_j)_{j=0}^m]$ of all non-negative hermitian $q \times q$ measures σ on Ω such that for all $j \in \{0, 1, \ldots, m\}$ the moments $\int_\Omega x^j \sigma(\mathrm{d}x)$ exist and fulfill

$$\int_\Omega x^j \sigma(\mathrm{d}t) = s_j.$$

Question

For which sequences $(s_j)_{j=0}^m$ the set $\mathcal{M}_{\succcurlyeq}^q[\Omega, (s_j)_{j=0}^m]$ is non-empty?

Outline

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Solvability for $\Omega = [\alpha, \beta]$ in terms of Hankel matrices

Let $\alpha < \beta$ and let $(s_j)_{j=0}^{m}$ be a sequence from $\mathbb{C}^{q\times q}$:

Theorem (e. g. Choque R./Dyukarev/Fritzsche/Kirstein 2006/07)

- $\mathcal{M}_{\succcurlyeq}^{q}[[\alpha,\beta],(s_j)_{j=0}^{2n+1}] \neq \emptyset$ *if and only if*

$$-\alpha H_n + K_n \succcurlyeq \mathbb{O} \qquad \text{and} \qquad \beta H_n - K_n \succcurlyeq \mathbb{O}.$$

- $\mathcal{M}_{\succcurlyeq}^{q}[[\alpha,\beta],(s_j)_{j=0}^{2n}] \neq \emptyset$ *if and only if*

$$H_n \succcurlyeq \mathbb{O} \qquad \text{and} \qquad -\alpha\beta H_{n-1} + (\alpha+\beta)K_{n-1} - G_{n-1} \succcurlyeq \mathbb{O}.$$

$\mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$: the set of all such Hausdorff moment sequences $(s_j)_{j=0}^{m}$ from $\mathbb{C}^{q\times q}$

$$H_n := [s_{j+k}]_{j,k=0}^{n}, \quad K_n := [s_{j+k+1}]_{j,k=0}^{n} \quad \text{and} \quad G_n := [s_{j+k+2}]_{j,k=0}^{n}.$$

Connection to truncated moment problems on the semi-axis $[\alpha, \infty)$ and $(-\infty, \beta]$

We have

$$\mathcal{M}^q_{\succcurlyeq}[[\alpha, \beta], (s_j)_{j=0}^{2n+1}] \neq \emptyset \iff \begin{cases} \mathcal{M}^q_{\succcurlyeq}[[\alpha, \infty), (s_j)_{j=0}^{2n+1}] \neq \emptyset \\ \text{and} \\ \mathcal{M}^q_{\succcurlyeq}[(-\infty, \beta], (s_j)_{j=0}^{2n+1}] \neq \emptyset \end{cases}$$

but only

$$\mathcal{M}^q_{\succcurlyeq}[[\alpha, \beta], (s_j)_{j=0}^{2n}] \neq \emptyset \Longrightarrow \begin{cases} \mathcal{M}^q_{\succcurlyeq}[[\alpha, \infty), (s_j)_{j=0}^{2n}] \neq \emptyset \\ \text{and} \\ \mathcal{M}^q_{\succcurlyeq}[(-\infty, \beta], (s_j)_{j=0}^{2n}] \neq \emptyset \end{cases}$$

with "$\not\Leftarrow$" for $n \geq 1$.

For all measures σ on $[0, 1]$ we have

$$0 \leq \int_{[0,1]} x^j \sigma(\mathrm{d}x) \leq \sigma([0, 1]), \qquad j = 0, 1, 2, \ldots$$

Example

$$\mu_1 := 9\delta_0 + \delta_5 \qquad \text{and} \qquad \mu_2 := \delta_{-4} + 9\delta_1$$

are representing measures for the sequence

$$s_0 := 10 \qquad s_1 := 5 \qquad s_2 := 25$$

on $[0, \infty)$ and $(-\infty, 1]$, respectively. But in view of $s_2 > s_0$, Problem $\mathrm{M}[[0, 1], (s_j)_{j=0}^2]$ has no solution.

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Associated block Hankel matrices

With the sequences

$$a_j := -\alpha s_j + s_{j+1}, \qquad b_j := \beta s_j - s_{j+1}, \qquad 0 \leq j \leq m-1,$$

and

$$c_j := -\alpha\beta s_j + (\alpha+\beta) s_{j+1} - s_{j+2}, \qquad 0 \leq j \leq m-2,$$

we have

$$H_n^{\mathrm{a}} := [a_{j+k}]_{j,k=0}^{n} = -\alpha H_n + K_n,$$
$$H_n^{\mathrm{b}} := [b_{j+k}]_{j,k=0}^{n} = \quad \beta H_n - K_n$$

and

$$H_n^{\mathrm{c}} := [c_{j+k}]_{j,k=0}^{n} = -\alpha\beta H_n + (\alpha+\beta) K_n - G_n.$$

Non-negative hermitian block matrices

For a block matrix $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ denote by

$$E/A = D - CA^{\dagger}B$$

the Schur complement of A in E.

Lemma

$E \succcurlyeq \mathbb{O}$ if and only if

$$A \succcurlyeq \mathbb{O}, \quad \mathcal{R}(B) \subseteq \mathcal{R}(A), \quad C = B^*, \quad \textit{and} \quad E/A \succcurlyeq \mathbb{O}.$$

- $\mathcal{R}(M) := \{Mx \colon x \in \mathbb{C}^q\}$ the *column space* of $M \in \mathbb{C}^{p \times q}$
- $M^{\dagger}$ the Moore-Penrose inverse of $M \in \mathbb{C}^{p \times q}$, i. e. the unique simultaneous solution $X \in \mathbb{C}^{q \times p}$ of

$$MXM = M, \quad XMX = X, \quad (MX)^* = MX, \quad (XM)^* = XM.$$

Four associated Schur complements

In view of the block partitions

$$H_n^{\mathrm{a}} = \left[\begin{array}{c:c} H_{n-1}^{\mathrm{a}} & * \\ \hdashline * & a_{2n} \end{array}\right] \qquad H_n^{\mathrm{b}} = \left[\begin{array}{c:c} H_{n-1}^{\mathrm{b}} & * \\ \hdashline * & b_{2n} \end{array}\right]$$

$$H_{n+1} = \left[\begin{array}{c:c} H_n & * \\ \hdashline * & s_{2n+2} \end{array}\right] \qquad H_n^{\mathrm{c}} = \left[\begin{array}{c:c} H_{n-1}^{\mathrm{c}} & * \\ \hdashline * & c_{2n} \end{array}\right],$$

denote by

$$\begin{aligned} \mathfrak{f}_{4k} &:= A_{2k} := H_k / H_{k-1} \\ \mathfrak{f}_{4k+1} &:= A_{2k+1} := H_k^{\mathrm{a}} / H_{k-1}^{\mathrm{a}} \\ \mathfrak{f}_{4k+2} &:= B_{2k+1} := H_k^{\mathrm{b}} / H_{k-1}^{\mathrm{b}} \\ \mathfrak{f}_{4k+3} &:= B_{2k+2} := H_k^{\mathrm{c}} / H_{k-1}^{\mathrm{c}} \end{aligned}$$

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Characterization of Hausdorff moment sequences

Let $(s_j)_{j=0}^{m}$ be a sequence from $\mathbb{C}^{q\times q}$:

Theorem

$(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ *if and only if* $\mathfrak{f}_j \succcurlyeq \mathbb{O}$ *for all* $j \in \{0, 1, 2, \dots, 2m\}$.

The mapping $(s_j)_{j=0}^{m} \mapsto (\mathfrak{f}_j)_{j=0}^{2m}$ is injective.

Example

The non-negative hermitian “matrices” $\mathfrak{f}_0 = \mathfrak{f}_1 = \mathfrak{f}_2 := 1$ don't belong to any sequence $(s_j)_{j=0}^{1}$ for $[\alpha, \beta] = [0, 1]$:

$$1 = \mathfrak{f}_0 = s_0 \qquad \text{and} \qquad 1 = \mathfrak{f}_1 = a_0 = -\alpha s_0 + s_1 = s_1$$

but

$$1 = \mathfrak{f}_2 = b_0 = \beta s_0 - s_1 = 0.$$

Extending sequences from $\mathcal{F}^{\succcurlyeq}_{q,m,\alpha,\beta}$ to $\mathcal{F}^{\succcurlyeq}_{q,m+1,\alpha,\beta}$

Let $(s_j)_{j=0}^{m} \in \mathcal{F}^{\succcurlyeq}_{q,m,\alpha,\beta}$ and let $s_{m+1} \in \mathbb{C}^{q\times q}$:

Theorem

$$(s_j)_{j=0}^{m+1} \in \mathcal{F}^{\succcurlyeq}_{q,m+1,\alpha,\beta} \iff A_{m+1} \succcurlyeq \mathbb{O} \text{ and } B_{m+1} \succcurlyeq \mathbb{O} \iff \mathfrak{a}_m \preccurlyeq s_{m+1} \preccurlyeq \mathfrak{b}_m$$

Here,

$$\mathfrak{a}_m := s_{m+1} - A_{m+1} \qquad \text{and} \qquad \mathfrak{b}_m := s_{m+1} + B_{m+1}$$

only depend on $s_0, s_1, \dots, s_m$.

Extendability

Given a sequence $(s_j)_{j=0}^{m} \in \mathcal{F}^{\succcurlyeq}_{q,m,\alpha,\beta}$, we have

$$s_j = \int_{[\alpha,\beta]} x^j \sigma(\mathrm{d}x), \qquad 0 \leq j \leq m,$$

for some representing $q \times q$ measure σ on $[\alpha, \beta]$. The next moment

$$s_{m+1} := \int_{[\alpha,\beta]} x^{m+1} \sigma(\mathrm{d}x)$$

of σ gives an extension $(s_j)_{j=0}^{m+1} \in \mathcal{F}^{\succcurlyeq}_{q,m+1,\alpha,\beta}$. In particular,

$$\mathfrak{a}_m \preccurlyeq s_{m+1} \preccurlyeq \mathfrak{b}_m.$$

Question

Can we see $\mathfrak{a}_m \preccurlyeq \mathfrak{b}_m$ without a representing $q \times q$ measure σ?

$\mathfrak{b}_{2n+1} - \mathfrak{a}_{2n+1}$ as Schur complement

For $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{\succcurlyeq}_{q,2n+1,\alpha,\beta}$ the difference $\mathfrak{b}_{2n+1} - \mathfrak{a}_{2n+1}$ is exactly the Schur complement $E/A = D - CA^\dagger B$ of the $nq \times nq$ block A in the matrix

$$(-\alpha H_n + K_n) H_n^\dagger (\beta H_n - K_n) =: E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

With

$$H_n^{\mathrm{a}} = -\alpha H_n + K_n \qquad \text{and} \qquad H_n^{\mathrm{b}} = \beta H_n - K_n$$

we have

$$H_n^{\mathrm{a}} + H_n^{\mathrm{b}} = (\beta - \alpha) H_n$$

and hence

$$E = (\beta - \alpha) H_n^{\mathrm{a}} (H_n^{\mathrm{a}} + H_n^{\mathrm{b}})^\dagger H_n^{\mathrm{b}} = (\beta - \alpha)(H_n^{\mathrm{a}} \mp H_n^{\mathrm{b}}).$$

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Parallel sum

Parallel resistors

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \qquad \longrightarrow \qquad R = \frac{R_1 R_2}{R_1 + R_2}$$

For $q \times q$ matrices $A \succcurlyeq \mathbb{O}$ and $B \succcurlyeq \mathbb{O}$ let

$$A \mp B := A(A+B)^\dagger B$$

be the parallel sum of A and B:

- $A \mp B = B \mp A$
- $\mathcal{R}(A \mp B) = \mathcal{R}(A) \cap \mathcal{R}(B)$
- $A \mp B \succcurlyeq \mathbb{O}$
- $A \mp B \preccurlyeq A$ and $A \mp B \preccurlyeq B$

Here, $\mathcal{R}(M) := \{Mx \colon x \in \mathbb{C}^q\}$ is the *column space* of $M \in \mathbb{C}^{p \times q}$.

$\mathfrak{b}_m - \mathfrak{a}_m$ as parallel sum

Let $E_1 \succcurlyeq \mathbb{O}$, $E_2 \succcurlyeq \mathbb{O}$, and $E_0 := E_1 \mp E_2$ be equally partitioned:

$$E_j = \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}, \qquad\qquad j = 0, 1, 2.$$

The corresponding Schur complements $E_j/A_j = D_j - C_j A_j^\dagger B_j$ are connected by $E_0/A_0 = (E_1/A_1) \mp (E_2/A_2)$.

Theorem

For $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ we have

$$\begin{aligned} \mathfrak{d}_m := \mathfrak{b}_m - \mathfrak{a}_m &= (\beta - \alpha)(A_m \mp B_m) \\ &= (\beta - \alpha)(B_m \mp A_m) \succcurlyeq \mathbb{O}. \end{aligned}$$

$\mathfrak{d}_m = \mathbb{O}$ if and only if $\mathcal{R}(s_m - \mathfrak{a}_{m-1}) \cap \mathcal{R}(\mathfrak{b}_{m-1} - s_m) = \{0\}$.

Four associated Schur complements

In view of the block partitions

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the corresponding Schur complements.

Central extension

HM-AM inequality

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \frac{a+b}{2} \qquad \longrightarrow \qquad 2(A \mp B) \preccurlyeq \frac{1}{2}(A+B)$$

In view of

$$\mathfrak{d}_{m+1} = (\beta - \alpha)(A_{m+1} \mp B_{m+1}) \quad \text{and} \quad \mathfrak{d}_m = A_{m+1} + B_{m+1}$$

we get for $(s_j)_{j=0}^{m+1} \in \mathcal{F}^{\succcurlyeq}_{q,m+1,\alpha,\beta}$ the inequality

$$\mathfrak{d}_{m+1} \preccurlyeq \frac{\beta-\alpha}{4}\mathfrak{d}_m$$

with "$=$" if and only if s_{m+1} is the midpoint $\frac{1}{2}(\mathfrak{a}_m + \mathfrak{b}_m)$ of the interval $[\mathfrak{a}_m, \mathfrak{b}_m]$ of possible extensions.

Decreasing lengths $\mathfrak{d}_k$ of extension intervals $[\mathfrak{a}_k, \mathfrak{b}_k]$

Let

$$s_j := \int_{[\alpha,\beta]} x^j \sigma(\mathrm{d}x), \qquad j = 0, 1, 2, \ldots$$

be the moments of some $q \times q$ measure σ on $[\alpha, \beta]$. Then

$$s_k \in [\mathfrak{a}_k, \mathfrak{b}_k], \qquad k = 0, 1, 2, \ldots,$$

and

$$\mathfrak{d}_0 \succcurlyeq \gamma^{-1}\mathfrak{d}_1 \succcurlyeq \gamma^{-2}\mathfrak{d}_2 \succcurlyeq \cdots \succcurlyeq \gamma^{-m}\mathfrak{d}_m \succcurlyeq \cdots \succcurlyeq \mathbb{O},$$

where $\mathfrak{d}_k := \mathfrak{b}_k - \mathfrak{a}_k$ and $\gamma := (\beta - \alpha)/4$.

Question

$$\lim_{k\to\infty} \gamma^{-k}\mathfrak{d}_k = ?$$

Hankel determinants

Suppose

$$\Delta_\ell := \det \mathfrak{d}_\ell > 0 \qquad \text{for all } \ell \in \{0, 1, 2, \ldots\}.$$

In view of

$$\mathfrak{d}_{k-1} = A_k + B_k \quad \text{and} \quad \mathfrak{d}_k = (\beta - \alpha) A_k (A_k + B_k)^{-1} B_k,$$

then

$$\begin{aligned} \Delta_{2j}\Delta_{2j+1} &= (\beta-\alpha)^q \det A_{2j+1} \det B_{2j+1} \\ &= (\beta-\alpha)^q \det(H_j^{a}/H_{j-1}^{a}) \det(H_j^{b}/H_{j-1}^{b}) \end{aligned}$$

and hence

$$\sqrt[n+1]{\Delta_0 \Delta_1 \cdots \Delta_{2n+1}} = (\beta-\alpha)^q \sqrt[n+1]{\det H_n^{a}} \sqrt[n+1]{\det H_n^{b}}.$$

Result of Szegő on Hankel forms

1918 Szegő A *Hankel*-féle formákról.
(Über die *Hankel*schen Formen.)

Let $0 < m \leq M$ and let $f\colon [\alpha, \beta] \to [m, M]$ be Riemann integrable with moments

$$s_j := \int_\alpha^\beta x^j f(x) \mathrm{d}x, \qquad j = 0, 1, 2, \ldots$$

Theorem (Szegő 1918)

$$\lim_{n\to\infty} \frac{\sqrt[n+1]{\det H_n}}{\gamma^{n+1}} = 2\pi \exp\left\{ \frac{1}{\pi} \int_\alpha^\beta \ln f(x) \frac{\mathrm{d}x}{\sqrt{(x-\alpha)(\beta-x)}} \right\}.$$

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the corresponding Schur complements.

Hidden dependence in $\mathfrak{f}_0, \mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{2m}$

For $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ we have the connections

$$\mathfrak{f}_1 + \mathfrak{f}_2 = A_1 + B_1 = a_0 + b_0 = (\beta - \alpha)s_0 = (\beta - \alpha)\mathfrak{f}_0$$

and

$$\begin{aligned}\mathfrak{f}_{2\ell+1} + \mathfrak{f}_{2\ell+2} = A_{\ell+1} + B_{\ell+1} &= \mathfrak{b}_\ell - \mathfrak{a}_\ell = \mathfrak{d}_\ell \\ &= (\beta - \alpha)(A_\ell \mp B_\ell) = (\beta - \alpha)(\mathfrak{f}_{2\ell-1} \mp \mathfrak{f}_{2\ell-2}).\end{aligned}$$

Theorem

$(s_j)_{j=0}^{m} \mapsto (\mathfrak{f}_j)_{j=0}^{2m}$ is one-to-one between $\mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ and the set of sequences $(f_j)_{j=0}^{2m}$ from $\mathbb{C}^{q\times q}$ with $f_j \succcurlyeq \mathbb{O}$ for $j \in \{0, \ldots, 2m\}$ and

$$(\beta - \alpha)f_0 = f_1 + f_2,$$
$$(\beta - \alpha)(f_{2\ell-1} \mp f_{2\ell}) = f_{2\ell+1} + f_{2\ell+2}, \qquad 1 \leq \ell \leq m - 1.$$

Outline

Canonical moments: Scalar case $q = 1$

The m-th moment space M_m on $[\alpha, \beta]$:

$$M_m := \left\{ \begin{bmatrix} \int_{[\alpha,\beta]} x^1 \mu(\mathrm{d}x) \\ \int_{[\alpha,\beta]} x^2 \mu(\mathrm{d}x) \\ \vdots \\ \int_{[\alpha,\beta]} x^m \mu(\mathrm{d}x) \end{bmatrix} : \mu([\alpha, \beta]) = 1 \right\} \subseteq \mathbb{R}^m$$

Describe position of next moment s_{m+1} in the interval $[\mathfrak{a}_m, \mathfrak{b}_m]$ with canonical moment p_{m+1} in $[0, 1]$:

$$s_{m+1} = \mathfrak{a}_m + p_{m+1}\mathfrak{d}_m \text{ with } p_{m+1} := \frac{s_{m+1} - \mathfrak{a}_m}{\mathfrak{b}_m - \mathfrak{a}_m} = \frac{A_{m+1}}{\mathfrak{d}_m} \in [0, 1]$$

1967 Skibinsky The range of the $(n + 1)$th moment for distributions on $[0, 1]$.

1997 Dette/Studden The theory of canonical moments with applications in statistics, probability, and analysis.

Canonical moments: Matrix case $q \geq 1$

2002 Dette/Studden Matrix measures, moment spaces and Favard's theorem for the interval $[0, 1]$ and $[0, \infty)$.

Let $(s_j)_{j=0}^{m}$ be in the interior of the moment space on $[0, 1]$:

Theorem (Dette/Studden 2002)

For

$$U_\ell := \mathfrak{d}_{\ell-1}^{-1} A_\ell \qquad \textit{and} \qquad V_\ell := \mathfrak{d}_{\ell-1}^{-1} B_\ell, \qquad 1 \leq \ell \leq m,$$

we have

$$\mathfrak{d}_m = s_0 U_1 V_1 U_2 V_2 \cdots U_m V_m \quad \textit{with} \quad U_\ell V_\ell = V_\ell U_\ell, \quad 1 \leq \ell \leq m.$$

Parametrization of $\mathcal{F}^{\succcurlyeq}_{q,m,\alpha,\beta}$

For $(s_j)_{j=0}^m \in \mathcal{F}^{\succcurlyeq}_{q,m,\alpha,\beta}$ we can, in view of $\mathfrak{d}_\ell \succcurlyeq \mathbb{O}$, define

$$\mathfrak{e}_0 := \mathfrak{f}_0 \qquad \text{and} \qquad \mathfrak{e}_j := \sqrt{\mathfrak{d}_{j-1}}^\dagger \mathfrak{f}_{2j} \sqrt{\mathfrak{d}_{j-1}}^\dagger, \qquad 1 \leq j \leq m.$$

Then, $\mathfrak{e}_j \succcurlyeq \mathbb{O}$ only depends on $s_0, s_1, \dots, s_j$.

Theorem

$(s_j)_{j=0}^m \mapsto (\mathfrak{e}_j)_{j=0}^m$ is one-to-one between $\mathcal{F}^{\succcurlyeq}_{q,m,\alpha,\beta}$ and the set $\mathcal{E}^{\succcurlyeq}_{q,m,\beta-\alpha}$ of sequences $(e_j)_{j=0}^m$ from $\mathbb{C}^{q\times q}$ with $\mathbb{O} \preccurlyeq e_0$ and $\mathbb{O} \preccurlyeq e_j \preccurlyeq \mathbb{P}_{j-1}$ for $j = 1, 2, \dots, m$ where $\mathbb{P}_\ell$ is the orthogonal projector onto the column space $\mathcal{R}(d_\ell)$ and

$$d_0 := (\beta - \alpha) e_0,$$

$$d_k := (\beta - \alpha)\sqrt{d_{k-1}}\sqrt{e_k}(\mathbb{P}_{k-1} - e_k)\sqrt{e_k}\sqrt{d_{k-1}}, 1 \leq k \leq m-1.$$

Let $(e_j)_{j=0}^{m} \in \mathcal{E}^{\succcurlyeq}_{q,m,\beta-\alpha}$ with corresponding Hausdorff moment sequence $(s_j)_{j=0}^{m} \in \mathcal{F}^{\succcurlyeq}_{q,m,\alpha,\beta}$:

Scalar case $q = 1$

We have $\mathbb{P}_\ell \in \{0; 1\}$ for $\ell = 0, 1, \ldots, m-1$:

$$e_0 > 0 \qquad e_1, e_2, \ldots, e_{k-1} \in (0, 1)$$

$$e_k \in \{0; 1\} \qquad e_{k+1} = e_{k+2} = \cdots = e_m = 0$$

Matrix case $q \geq 1$

- If $e_j = \mathbb{O}$ or $\mathbb{P}_{j-1}$, then $s_j = \mathfrak{a}_{j-1}$ or $\mathfrak{b}_{j-1}$, resp., and $\mathfrak{d}_j = \mathbb{O}$.
- If $e_j = \frac{1}{2}\mathbb{P}_{j-1}$, then $s_j = \frac{1}{2}(\mathfrak{a}_{j-1} + \mathfrak{b}_{j-1})$ and $\mathfrak{d}_j$ is maximal.
- If $d_k = \mathbb{O}$, then $\mathbb{P}_{j-1} = \mathbb{O}$ and $e_j = \mathbb{O}$ for $j = k+1, \ldots, m$.
- If $\det d_k \neq 0$, then $\mathbb{P}_{j-1} = \mathbb{I}$ and $e_j \in (\mathbb{O}, \mathbb{I})$ for $j = 1, \ldots, k$.

Thank you for your kind attention.

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