Spectral methods to compute a solution to some H^∞ interpolation problems

A. E. Frazho

The talk is mainly my approach on using FFT and spectral methods to compute solutions to H^∞ problems. It bypasses many state space methods. Some of the methods are standard and nothing new is claimed.

A review of FFT

Consider a trigonometric polynomial

$$p(e^{i\omega}) = \sum_k a_k e^{-i\omega k}$$

The DFT is the Vandermonde matrix $\mathfrak{F}_\nu$ on $\mathbb{C}^\nu$ containing $\{e^{-i\omega_j}\}$

$$\begin{bmatrix} p(e^{i\omega_0}) \\ p(e^{i\omega_1}) \\ \vdots \\ p(e^{i\omega_{\nu-2}}) \\ p(e^{i\omega_{\nu-1}}) \end{bmatrix} = \mathfrak{F}_\nu \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{-2} \\ a_{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{-2} \\ a_{-1} \end{bmatrix} = \mathfrak{F}_\nu^{-1} \begin{bmatrix} p(e^{i\omega_0}) \\ p(e^{i\omega_1}) \\ \vdots \\ p(e^{i\omega_{\nu-2}}) \\ p(e^{i\omega_{\nu-1}}) \end{bmatrix}$$

Here $\{\omega_j\}_0^{\nu-1}$ is ν evenly spaces points in $[0, 2\pi)$. If $\nu = 2^n$ the computation is $\nu \log(\nu)$ and called FFT. The inverse DFT is

$$\boxed{\mathfrak{F}_\nu^{-1} = \frac{1}{\nu}\mathfrak{F}_\nu^*}$$

Approximating Fourier series and H^2 functions

Let $\mathfrak{F} : \ell^2_+ \to L^2$ be the Fourier transform:

$$f(e^{i\omega}) = \sum \alpha_k e^{-ik\omega} = \mathfrak{F} \begin{bmatrix} \cdots & \alpha_{-1} & \alpha_0 & \alpha_1 & \cdots \end{bmatrix}^\top$$

For continuous f evaluate $f(e^{i\omega})$ on 2^n points of $[0, 2\pi)$: take ifft

$$a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{-2} \\ a_{-1} \end{bmatrix} = \mathfrak{F}_\nu^{-1} \begin{bmatrix} f(e^{i\omega_0}) \\ f(e^{i\omega_1}) \\ \vdots \\ f(e^{i\omega_{\nu-2}}) \\ f(e^{i\omega_{\nu-1}}) \end{bmatrix}$$

$$a_k \approx \alpha_k = \frac{1}{2\pi} \int_0^{2\pi} e^{ik\omega} f(e^{i\omega}) d\omega$$ **ifft computes Reiman intergal**

$\|f\|_2 \approx \|a\|_{\mathbb{C}^\nu}$ and $\|f\|_\infty \approx \|f(e^{i\omega_j})\|_{\infty,\mathbb{C}^\nu}$

$f \in H^2 \iff$ **the bottom half of** a **equals zero**

Operator theory uses $e^{ik\omega}$. Matlab, engineers $e^{-ik\omega}$ $(\lambda = z^{-1})$.

Transfer functions $G(z) = \frac{n(z)}{d(z)} = \sum a_k z^{-k}$

For example,

$$G(z) = \frac{0.84z^2 + 1.88z + 0.66}{z^4 + 0.44z^3 - 0.43z^2 - 0.056z + 0.014}$$
$$= \frac{0.84}{z^2} + \frac{1.5}{z^3} + \frac{0.35}{z^5} + \frac{0.53}{z^6} + \cdots$$

Matlab commands: $2^{16} = 65536$ is overkill.

$$g = \text{fft}([0, 0, 0.84, 1.88, 0.66], 2^{16})./$$
$$\text{fft}([1, 0.44, -0.43, -0.056, 0.014], 2^{16});$$
$$m = \text{real}(\text{ifft}(g));$$
$$m(1:6) = \begin{bmatrix} 0 & 0 & 0.84 & 1.5 & 0.35 & 0.53 \end{bmatrix}$$

$\text{norm}(m(2^{15} : 2^{16}) = 3.64 \times 10^{-16}$; **Therefore $G(z)$ is stable**

$\text{norm}(g, \text{inf}) = 3.47 = \|G\|_\infty$ and $\text{norm}(m) = 1.87 = \|G\|_2$

The Nyquist and Magnitude plot

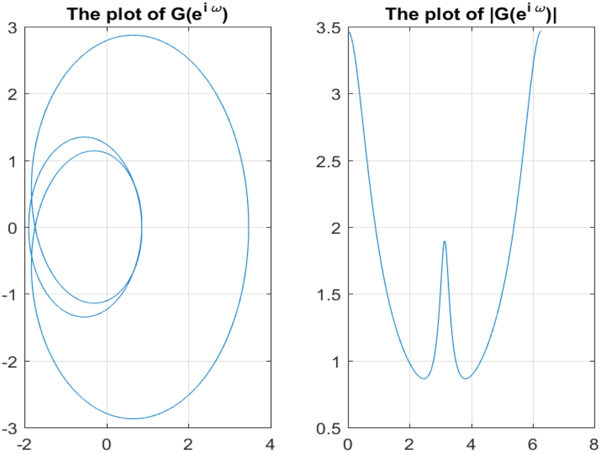


Figure: The plot of $G(e^{i\omega})$ and $|G(e^{i\omega})|$; winding number $= 3$.

Stability of polynomials

The polynomial $p(z)$ is stable if and only if $\frac{z^n}{p(z)}$ is in H^2. The polynomial

$$
\begin{aligned}
p(z) &= (z-2)\prod_{k=1}^{5}\left(z-\frac{k}{10}\right) \\
&= z^6 - 3.5z^5 + 3.85z^4 - 1.925z^3 + 0.4774z^2 - 0.056z + 0.0024
\end{aligned}
$$

is unstable. Matlab commands: ($2^{20} = 1048576$ is way overkill)

$$
\begin{aligned}
&p = \text{poly}([(1:5)/10, 2]);\, f = 1./\text{fft}(p, 2^{20}); \\
&m = \text{ifft}(f);\, \text{norm}(m(2^{19}:2^{20})) = 1.324 \ \ (\textbf{unstable}) \\
&\text{norm}(\text{ifft}(f)) = 2.3625 = \|f\|_2 \\
&(\text{ifft}(f), \text{inf}) = 1.2933 = \|f\|_\infty
\end{aligned}
$$

Hankel approximation: Classical Kalman-Ho

Consider a rational stable

$$G(z) = \frac{n(z)}{d(z)} = \sum_{n=0}^{\infty} a_n z^{-n}$$

The corresponding Hankel matrix

$$H = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ a_3 & a_4 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}$$

rank H = dim of minimal realization of

$$G(z) = D + C(zI - A)^{-1}B$$

Using the finite partition of H with the svd the Kalman-Ho algorithm computes $\{A, B, C, D\}$.

Hankel approximation: example

Consider the non rational function

$$f(z) = e^{\frac{2}{z} + \frac{1}{z^2}}$$

Using the FFT with Kalman-Ho

$$f(z) \approx g = \frac{z^4 + 0.5039z^3 + 0.1091z^2 + 0.0123z + 0.0006292}{z^4 - 0.4961z^3 + 0.1052z^2 - 0.01152z + 0.0005635}$$

$\|f\|_\infty = 20.0855$ and $\|f\|_2 = 6.9435$ and $\|f - g\|_\infty = 0.0082$

As expected, $g(z)$ is outer. Matlab commands:

$$f = \exp(\mathrm{fft}([0, 2, 1], 2^{16}));$$
$$m = \mathrm{real}(\mathrm{ifft}(f)); [a, b, c, d] = \mathrm{kalho}(m(1 : 700), .01);$$
$$[n, dd] = \mathrm{ss2tf}(a, b, c, d); g = \mathrm{fft}(n, 2^{16})./\mathrm{fft}(dd, 2^{16});$$
$$\mathrm{norm}(f, inf) = 20.085 = \|f\|_\infty$$
$$\mathrm{norm}(m) = 6.94 = \|f\|_2,\ \|f - g\|_\infty = 0.0082$$

Toeplitz matrices

If $r = \sum_{-\infty}^{\infty} e^{-ik\omega} \in L^\infty$, then T_r is the Toeplitz matrix:

$$T_r = \begin{bmatrix} r_0 & r_{-1} & r_{-2} & \cdots \\ r_1 & r_0 & r_{-1} & \cdots \\ r_2 & r_1 & r_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ on } \ell^2_+ \quad \text{and} \quad \|T_r\| = \|r\|_\infty$$

If $\theta = \sum_0^\infty \theta_k z^{-k}$ is in H^∞, then

$$T_\theta = \begin{bmatrix} \theta_0 & 0 & 0 & \cdots \\ \theta_1 & \theta_0 & 0 & \cdots \\ \theta_2 & \theta_1 & \theta_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ on } \ell^2_+$$

- $\theta \in H^\infty$ is *outer* if $\overline{\text{ran}(T_\theta)} = \ell^2_+$.
- θ is an *outer spectral factor* for T_r if θ is outer and $T_r = T_\theta^* T_\theta$
- If $T_r \gg 0$, then T_r admits a unique outer spectral factor θ.

Outer spectral factorization

Let $T_r \gg 0$ be Toeplitz. Solve

$$T_r \begin{bmatrix} 1 & a_1 & a_2 & \cdots \end{bmatrix}^\top = \begin{bmatrix} e & 0 & 0 & \cdots \end{bmatrix}^\top \text{ and } \theta(z) = \frac{\sqrt{e}}{\sum_0^\infty a_k z^{-k}}$$

θ is *outer*, $T_r = T_\theta^* T_\theta$. Let T_n the upper $n \times n$ corner of T_r. Solve

$$T_n \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix}^\top = \begin{bmatrix} e & 0 & 0 & \cdots & 0 \end{bmatrix}^\top$$
$$\theta_n(z) = \frac{\sqrt{e} z^{n-1}}{z^{n-1} + a_1 z^{n-2} \cdots + \cdots a_{n-1}}$$
$$\theta(z) = \lim_{n\to\infty} \theta_n(z)$$

- $T_n = \left[T_{\theta_n}^* T_{\theta_n} \right]_n$ and $\theta_n(z)$ is outer
- $\theta_n(z)$ can be computed fast by Levinson, $n = 10000$ is easy.
- If r is rational, then $M_{deg}(\theta) < \infty$.
- $M_{deg}(\theta_n) = n - 1 \to \infty$. The Kalman-Ho finds θ from θ_n.
- The Matrix case is similar.

Inner outer factorization $g = g_i g_o$: an example

The matrix case is similar. Pole-zero cancelation unstable.

- Form the Toeplitz T_r from $|g|^2$. Use Levinson to find $\theta_n \approx g_o$.
- Compute $g_i = \frac{g}{g_o}$.
- Apply Kalman-Ho to find state space realizations.

$$g(z) = \frac{0.84z^2 + 1.88z + 0.66}{z^4 + 0.44z^3 - 0.43z^2 - 0.056z + 0.014}$$

$$g_o(z) = \frac{1.5z^4 + 1.5z^3 + 0.37z^2}{z^4 + 0.44z^3 - 0.43z^2 - 0.0555z + 0.014} \quad \text{and} \quad g_i(z) = \frac{0.557z + 1}{z^2(1 + 0.557z)}$$

$$g = \text{fft}(\text{num}, 2^{16})./\text{fft}(\text{den}, 2^{16}); \;\; r = \text{real}(\text{ifft}(\text{abs}(g).^2));$$

$$[a, e] = \text{levinson}(r(1:1000));$$

$$go = \text{sqrt}(e)./\text{fft}(a, 2^{16}); gi = g./go;$$

$$mo = \text{real}(\text{ifft}(go)); mi = \text{real}(ifft(gi));$$

$$[ao, bo, co, dko] = \text{kalho}(mo(1:700));$$

$$[no, do] = \text{ss2tf}(ao, bo, co, dko); Go = \text{tf}(no, do)$$

$$[ai, bi, ci, dki] = \text{kalho}(mi(1:700));$$

$$[ni, di] = \text{ss2tf}(ai, bi, ci, dki); \;\; Gi = \text{tf}(ni, di)$$

A classical outer factorization formula: scalar case

$$g_o(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{z+e^{i\omega}}{z-e^{i\omega}} \ln(|g(e^{i\omega})|d\omega\right)$$

Change the form for FFT. Notice that

$$2\ln|g(e^{i\omega})| = \sum_{n=-\infty}^{\infty} a_n e^{-in\omega} = h + \overline{h} \text{ and } h = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-in\omega}$$

Claim: $g_o(z) = e^h$ which is suitable for FFT

$$|g(e^{i\omega})|^2 = e^{2\ln|g(e^{i\omega})|} = e^h\overline{e^h} = g_o\overline{g}_o$$

Same example: $g(z) = \frac{0.84z^2+1.88z+0.66}{z^4+0.44z^3-0.43z^2-0.056z+0.014}$

$$g = \text{fft}(\text{num}, 2^{16})./\text{fft}(\text{den}, 2^{16}); \quad a = \text{ifft}(2\log(\text{abs}(g)));$$

$$gc = \exp(\text{fft}([a(1)/2, a(2:2^{15})], 2^{16})); \% \text{ this } \sim e^h$$

$$\text{norm}(gc - go, inf) = 8.9543\times 10^{-15} = \|g_c - g_{lev}\|_\infty$$

$$g_o(z) = \frac{1.5z^4 + 1.5z^3 + 0.37z^2}{z^4 + 0.44z^3 - 0.43z^2 - 0.0555z + 0.014}$$

The band formula: Toeplitz extension

Many matrix H^∞ interpolation problems similar calculations.

Let $T_{n+1} \gg 0$ be Toeplitz on $\mathbb{C}^{n+1}$ with entries $\{r_j\}_0^n$.

Carathéodory-Toeplitz interpolation: Find all strictly positive Toeplitz extensions $T_r \sim \{r_j\}$ of T_{n+1}.

The band method solution:

$$T_{n+1} \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_n \end{bmatrix}^\top = \begin{bmatrix} e & 0 & 0 & \cdots & 0 \end{bmatrix}^\top$$
$$\sqrt{e}u = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_n z^{-n}$$

All strictly positive Toeplitz extensions $T_r \sim \{r_j\}$ of T_{n+1} are

$$r = \frac{1 - |g|^2}{|z^{-n}\overline{u}g + u|^2} = \sum_{k=-\infty}^{\infty} r_k z^{-k} = |\theta|^2 \qquad (g \in \mathcal{S}_\circ)$$

How to compute $\{r_k\}$ and outer θ such that $T_r = T_\theta^* T_\theta$.

A band method example:

Let T_4 the Toeplitz from $\{10, 9, 8, 6\}$ and

$$g(z) = \frac{1.275z - 1.797}{0.9z^3 - 1.068z^2 - 0.3655z + 0.5393} \quad \text{and} \quad \|g\|_\infty = .9$$

Some Matlab commands: $2^{17} = 131072$ is overkill.

$[a, e] = \text{levinson}([10, 9, 8, 6]); u = \text{fft}(a, 2^{17})./\text{sqrt}(e);$

$r = (1 - \text{abs}(g).^2)./(\text{abs}(\text{conj}(u).*g.*\text{fft}([0, 0, 0, 1], 2^{17}) + u).^2);$

$r_j = real(ifft(r)); r_j(1:5) = \begin{bmatrix} 10 & 9 & 8 & 6 & 4.25 \end{bmatrix}$

$[q, e] = \text{levinson}(r_j(1:2000)); \theta = \text{sqrt}(e)./\text{fft}(q, 2^{17});$

$norm(r - abs(\theta).^2, \text{inf}) = 2.2021 \times 10^{-10} = \|r - |\theta|^2\|_\infty$

Applying Kalman-Ho to ifft(θ) yields the outer factor

$$\theta(z) = \frac{1.106z^6 - 1.335z^5 - 0.4449z^4 + 0.677z^3}{z^6 - 2.103z^5 + 0.1902z^4 + 2.128z^3 - 1.041z^2 - 0.5014z + 0.328}$$

Optimal classical Nevanlinna-Pick interpolation

Consider a stable A and controllability operators.

$$W = \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} : \ell_+^2 \to \mathcal{X}$$

$$\widetilde{W} = \begin{bmatrix} \widetilde{B} & A\widetilde{B} & A^2\widetilde{B} & \cdots \end{bmatrix} : \ell_+^2 \to \mathcal{X}$$

$$WW^* = P = APA^* + BB^*$$

$$\widetilde{W}\widetilde{W}^* = \widetilde{P} = A\widetilde{P}A^* + \widetilde{B}\widetilde{B}^*$$

Find

$$\delta = \inf\left\{\|f\|_\infty : f \in H^\infty \text{ and } WT_f = \widetilde{W}\right\}.$$

Solution: Compute $P^{-1}\widetilde{P}\mathbf{x} = \lambda_{max}\mathbf{x}$. Then

$$\theta(z) = \frac{\lambda_{max} B^*(zI - A^*)^{-1}\mathbf{x}}{\widetilde{B}^*(zI - A^*)^{-1}\mathbf{x}} \quad \text{and} \quad \delta = \sqrt{\lambda_{max}}$$

Pole zero cancellation is numerically sensitive.

$\frac{\theta(z)}{\sqrt{\lambda_{max}}}$ is a Blaschke product

Optimal Nevanlinna-Pick interpolation: Example

$$A = \begin{bmatrix} 0.7512 & 0.1050 & 0.1775 & -0.0996 \\ -0.0292 & 0.5521 & 0.1321 & -0.0696 \\ 0.1395 & -0.0760 & 0.6226 & -0.1585 \\ -0.1792 & -0.0795 & -0.0378 & 0.6881 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3545 \\ 0.4106 \\ 0.9843 \\ 0.9456 \end{bmatrix} \text{ and } \widetilde{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

In this case, $\delta = \sqrt{\lambda_{max}} = 16.51$ and

$$\theta(z) = 16.51 \frac{-1.878z^3 + 17.57z^2 - 32.15z + 16.51}{16.51z^3 - 32.15z^2 + 17.57z - 1.878}$$

Some Matlab commands: First compute P, $\widetilde{P}$, $\mathbf{x}$ and λ_{max}

$[n, d] = \text{ss2tf}(A', \mathbf{x}, B', 0); g = \text{fft}(n, 2^{17})./\text{fft}(d, 2^{17});$

$[n, d] = \text{ss2tf}(A', \mathbf{x}, \widetilde{B}', 0); f = \text{fft}(n, 2^{17})./\text{fft}(d, 2^{17});$

$\theta = \lambda_{max} * g./f; m = \text{real}(\text{ifft}(\theta));$ Check answer: $WT_\theta = \widetilde{W}$

$s = b * m(1);$ for $k = 1 : 1000; s = s + a^k * b * m(k+1);$ end;

$s = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^\top = \widetilde{B}$

%Run Kalman-Ho on $m(1 : 800)$ to find $\theta(z)$.

The H^2 distance to the Schur functions

Let f be in H^2 and $\mathcal{S} = \{g \in H^\infty : \|g\|_\infty \leq 1\}$. Try to compute

$$\delta = \inf\{\|f - g\|_2 : g \in \mathcal{S}\}$$

Use alternating projections. Consider the convex set

$$\mathcal{C} = \{h \in L^\infty : \|h\|_\infty \leq 1\}$$

Then $S = \mathcal{C} \cap H^2$. Let P_+ be the orthogonal projection onto H^2 and $P_\mathcal{C}$ the orthogonal projection onto $\mathcal{C}$. Notice that

$$\begin{aligned}(P_\mathcal{C} f)(e^{i\omega}) &= \frac{f(e^{i\omega})}{|f(e^{i\omega})|} \quad \text{if } |f(e^{i\omega})| > 1 \\ &= f(e^{i\omega}) \qquad \text{if } |f(e^{i\omega})| \leq 1\end{aligned}$$

$$P_+ \sum_{k=-\infty}^{\infty} a_k e^{-ik\omega} = \sum_{k=0}^{\infty} a_k e^{-ik\omega}$$

By alternating projections $(P_+ P_\mathcal{C})^n f \to g \in \mathcal{S}$.

Dykstra algorithm $(P_+, P_\mathcal{C}) \to P_\mathcal{S}$

Example: H^2 distance to the Schur functions

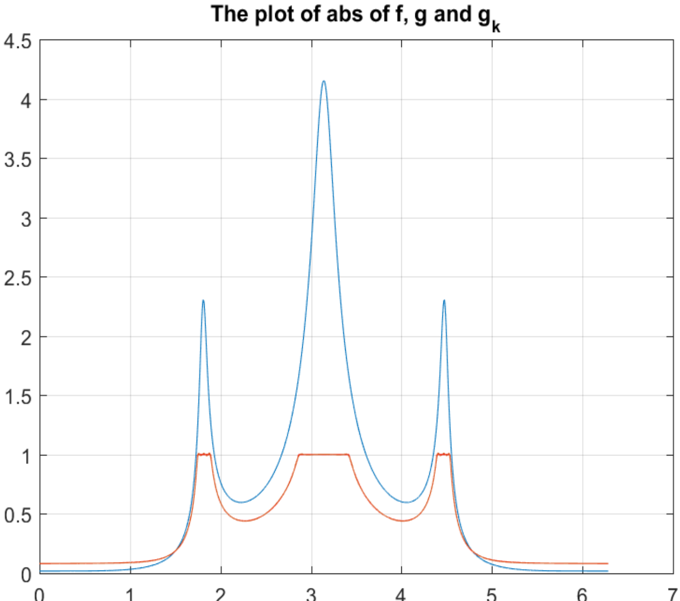
$$f = \frac{0.1837z^3 - 0.1198z^2 + 0.04688z + 0.008548}{z^4 + 1.859z^3 + 2.013z^2 + 1.499z + 0.4269}$$

$$g_k = \frac{0.11z^{10} + 0.57z^9 + 1.29z^8 + 2.08z^7 + 2.46z^6 + 2.23z^5 + 1.61z^4 + 0.9z^3 + 0.38z^2 + 0.13z + 0.02}{z^{10} + 4.59z^9 + 11.71z^8 + 20.96z^7 + 28.06z^6 + 29.13z^5 + 23.67z^4 + 14.79z^3 + 6.88z^2 + 2.18z + 0.35}$$

Matlab commands:

$$f = \text{fft}(\text{num}, 2^{15})./\text{fft}(\text{den}, 2^{15});$$
$$p = \text{zeros}(\text{size}(f)); q = p; x = f;$$
$$\text{for } k = 1 : 800; y = x + p; \text{for } j = 1 : 2^{15};$$
$$\text{if abs}(y(j)) > 1; y(j) = y(j)/\text{abs}(y(j)); \text{end;end}$$
$$p = x + p - y; m = \text{ifft}(y + q);$$
$$x = \text{fft}(m(1 : 2^{14}), 2^{15}); q = y + q - x; \text{end}$$
$$norm(ifft(f - x)) = 0.6879 = \|f - x\|_2$$
$$norm(f - x, inf) = 3.1522 = \|f - x\|_\infty$$
$$\%\text{Run Kalman-Ho on ifft(x)}; \ \|x - x_{kh}\|_\infty = 0.0239$$

The plot



Example: H^2 distance to the Schur functions

$$f = 1 + \frac{1}{z} \text{ and } g_k = \frac{0.5657z^2 + 0.8042z + 0.09876}{z^2 + 0.3556z + 0.1278} \text{ and } \|P_S f - g\| = 0.0451$$