Banach algebras of convolution type operators with *PSO* data on weighted Lebesgue spaces

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IWOTA 2017, TU Chemnitz,
Chemnitz, Germany, August 14-18, 2017

Weighted Lebesgue spaces

A measurable function $w : \mathbf{R} \to [0, \infty]$ is called a weight if the preimage $w^{-1}(\{0, \infty\})$ has measure zero. For $1 < p < \infty$, a weight w belongs to the Muckenhoupt class $A_p(\mathbf{R})$ if

$$c_{p,w} := \sup_I \left(\frac{1}{|I|} \int_I w^p(x)dx \right)^{1/p} \left(\frac{1}{|I|} \int_I w^{-q}(x)dx \right)^{1/q} < \infty,$$

where $1/p + 1/q = 1$, and supremum is taken over all intervals $I \subset \mathbf{R}$ of finite length $|I|$.

Given $1 < p < \infty$ and $w \in A_p(\mathbf{R})$, we consider the weighted Lebesgue space $L^p(\mathbf{R}, w)$ equipped with the norm

$$\|f\|_{L^p(\mathbf{R},w)} := \left(\int_{\mathbf{R}} |f(x)|^p w^p(x)dx \right)^{1/p}.$$

Let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators acting on a Banach space X.

The space $SO^\diamond$

For a continuous function $f : \mathbf{R} \to \mathbf{C}$ and a set $I \subset \mathbf{R}$, let

$$\mathrm{osc}\,(f, I) = \sup\{|f(t) - f(s)| : t, s \in I\}.$$

Given $\lambda \in \dot{\mathbf{R}} := \mathbf{R} \cup \{\infty\}$, we denote by SO_λ the C^*-algebra of functions slowly oscillating at λ,

$$SO_\infty := \Big\{ f \in C_b(\dot{\mathbf{R}} \setminus \{\infty\}) : \lim_{x\to+\infty} \mathrm{osc}(f, [-2x, -x] \cup [x, 2x]) = 0 \Big\},$$

$$SO_\lambda := \Big\{ f \in C_b(\dot{\mathbf{R}} \setminus \{\lambda\}) : \lim_{x\to 0} \mathrm{osc}(f, \lambda + ([-2x, -x] \cup [x, 2x])) = 0 \Big\}$$

for $\lambda \in \mathbf{R}$, where $C_b(\dot{\mathbf{R}} \setminus \{\lambda\}) = C(\dot{\mathbf{R}} \setminus \{\lambda\}) \cap L^\infty(\mathbf{R})$. Let $SO^\diamond$ be the minimal C^*-subalgebra of $L^\infty(\mathbf{R})$ that contains all the C^*-algebras SO_λ with $\lambda \in \dot{\mathbf{R}}$, and therefore contains $C(\dot{\mathbf{R}})$.
Let $C^n(\mathbf{R})$ be the set of all n times continuously differentiable functions $a : \mathbf{R} \to \mathbf{C}$. Let $\mathcal{F} : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ denote the Fourier transform,

$$(\mathcal{F}f)(x) := \int_{\mathbf{R}} f(t) e^{itx} dt, \; x \in \mathbf{R}.$$

Fourier multipliers on $L^p(\mathbb{R}, w)$

A function $a \in L^\infty(\mathbf{R})$ is called a Fourier multiplier on $L^p(\mathbf{R}, w)$ if the convolution operator $W^0(a) := \mathcal{F}^{-1} a \mathcal{F}$ maps the dense subset $L^2(\mathbf{R}) \cap L^p(\mathbf{R}, w)$ of $L^p(\mathbf{R}, w)$ into itself and extends to a bounded linear operator on $L^p(\mathbf{R}, w)$. Let $M_{p,w}$ stand for the Banach algebra of all Fourier multipliers on $L^p(\mathbf{R}, w)$ equipped with the norm $\|a\|_{M_{p,w}} := \|W^0(a)\|_{\mathcal{B}(L^p(\mathbf{R},w))}$.

To define slowly oscillating functions in $M_{p,w}$, we need such fact.

Theorem (Yu. K./Juan Loreto-Hernández)

If $a \in C^3(\mathbf{R} \setminus \{0\})$ and $\|D^\gamma a\|_{L^\infty(\mathbf{R})} < \infty$ for all $\gamma = 0, 1, 2, 3$, where $(Da)(x) = xa'(x)$ for $x \in \mathbf{R}$, then the convolution operator $W^0(a) = \mathcal{F}^{-1} a \mathcal{F}$ is bounded on every weighted Lebesgue space $L^p(\mathbf{R}, w)$ with $1 < p < \infty$ and $w \in A_p(\mathbf{R})$, and

$$\|W^0(a)\|_{\mathcal{B}(L^p(\mathbf{R},w))} \le c_{p,w} \max\left\{\|D^\gamma a\|_{L^\infty(\mathbf{R})} : \gamma = 0, 1, 2, 3\right\} < \infty,$$

where the constant $c_{p,w} \in (0, \infty)$ depends only on p and w.

Banach algebras $SO^\diamond_{p,w}$ and $PSO^\diamond_{p,w}$

For $\lambda \in \dot{\mathbf{R}}$, consider the commutative Banach algebras
$$SO^3_\lambda := \left\{ a \in SO_\lambda \cap C^3(\mathbf{R} \setminus \{\lambda\}) : \lim_{x\to\lambda} (D^\gamma_\lambda a)(x) = 0,\ \gamma = 1,2,3 \right\}$$
with norm $\|a\|_{SO^3_\lambda} := \max\left\{ \|D^\gamma_\lambda a\|_{L^\infty(\mathbf{R})} : \gamma = 0,1,2,3 \right\}$, where $(D_\lambda a)(x) = (x-\lambda)a'(x)$ for $\lambda \in \mathbf{R}$ and $(D_\lambda a)(x) = xa'(x)$ if $\lambda = \infty$. Then $SO^3_\lambda \subset M_{p,w}$. Let $SO_{\lambda,p,w}$ denote the closure of SO^3_λ in $M_{p,w}$, and let $SO^\diamond_{p,w}$ be the Banach subalgebra of $M_{p,w}$ generated by all algebras $SO_{\lambda,p,w}$ ($\lambda \in \dot{\mathbf{R}}$). By the continuous embedding $M_{p,w} \subset M_2 = L^\infty(\mathbf{R})$, we get $SO^\diamond_{p,w} \subset SO^\diamond$.

Let $V(\mathbf{R})$ be the Banach algebra of all functions $a : \overline{\mathbf{R}} \to \mathbf{C}$ with finite total variation $V(a)$ and the norm $\|a\|_V = \|a\|_\infty + V(a)$. By Stechkin's inequality, $V(\mathbf{R}) \subset M_{p,w}$.

Let $PSO^\diamond_{p,w} := \operatorname{alg}(SO^\diamond_{p,w}, PC_{p,w})$ be the Banach subalgebra of $M_{p,w}$ generated by the algebras $SO^\diamond_{p,w}$ and $PC_{p,w}$, where $PC_{p,w}$ is the closure in $M_{p,w}$ of the set $PC \cap V(\mathbf{R})$ of all piecewise continuous functions in $V(\mathbf{R})$.

Maximal ideal spaces of $SO^{\diamond}_{p,w}$ and $M(PSO^{\diamond}_{p,w})$

Identifying the points $\lambda \in \dot{\mathbb{R}}$ with the evaluation functionals δ_λ on $\dot{\mathbb{R}}$ for $f \in C(\dot{\mathbb{R}})$, $\delta_\lambda(f) = f(\lambda)$, we infer that the maximal ideal space $M(SO^\diamond)$ of $SO^\diamond$ has the form $M(SO^\diamond) = \bigcup_{\lambda\in\dot{\mathbb{R}}} M_\lambda(SO^\diamond)$ where $M_\lambda(SO^\diamond) = \left\{\xi \in M(SO^\diamond) : \xi|_{C(\dot{\mathbb{R}})} = \delta_\lambda\right\}$ are the fibers of $M(SO^\diamond)$ over $\lambda \in \dot{\mathbb{R}}$. One can show that for every $\lambda \in \dot{\mathbb{R}}$,

$$M_\lambda(SO^\diamond) = M_\lambda(SO_\lambda) = M_\infty(SO_\infty) = (\operatorname{clos}_{SO^*_\infty}\mathbb{R}) \setminus \mathbb{R},$$

where $\operatorname{clos}_{SO^*_\infty}\mathbb{R}$ is the weak-star closure of $\mathbb{R}$ in SO^*_∞.

Theorem

If $p \in (1,\infty)$ and $w \in A_p(\mathbb{R})$, then the maximal ideal spaces of $SO^\diamond_{p,w}$ and $SO^\diamond$ coincide as sets, $M(SO^\diamond_{p,w}) = M(SO^\diamond)$.

Lemma

If $p \in (1,\infty)$ and $w \in A_p(\mathbb{R})$, then the maximal ideal space $M(PSO^\diamond_{p,w})$ of $PSO^\diamond_{p,w}$ can be identified with $M(SO^\diamond) \times \{0,1\}$.

Compactness

Let $p \in (1, \infty)$, $w \in A_p(\mathbf{R})$, and let $\mathcal{K}_{p,w}$ be the ideal of compact operators in the Banach algebra $\mathcal{B}_{p,w} = \mathcal{B}(L^p(\mathbf{R}, w))$. Let us study the Banach subalgebra $\mathfrak{A}_{p,w}$ of $\mathcal{B}_{p,w}$ generated by all operators aI ($a \in PSO^\diamond$) and $W^0(b)$ ($b \in PSO^\diamond_{p,w}$).

Theorem (Yu. K./Iván Loreto-Hernández)

If either $a \in PSO^\diamond$ and $b \in SO^\diamond_{p,w}$, or $a \in SO^\diamond$ and $b \in PSO^\diamond_{p,w}$, or $a \in \operatorname{alg}(C(\overline{\mathbf{R}}), SO_\infty)$ and $b \in \operatorname{alg}(C_{p,w}(\overline{\mathbf{R}}), SO_{\infty,p,w})$, then $[aI, W^0(b)] \in \mathcal{K}_{p,w}$.

It suffices to prove the compactness of $[aI, W^0(b)]$ on the space $L^2(\mathbf{R})$. Let $a \in SO^\diamond \subset VMO$ and $b \in SO^3_\infty$. Then $W^0(b)$ is a bounded on $L^2(\mathbf{R})$ Calderón-Zygmund operator. Then

$$\|[aI, W^0(b)]\|_{\mathcal{B}(L^2(\mathbf{R}))} \le C\|a\|_{BMO}$$

for every $a \in BMO(\mathbf{R})$. On the other hand, VMO is the closure in $BMO(\mathbf{R})$ of the set $C(\dot{\mathbf{R}})$, and hence $[aI, W^0(b)] \in \mathcal{K}(L^2(\mathbf{R}))$.

A Banach algebra $\mathcal{Z}_{p,w}$

Consider the Banach algebra

$$\mathcal{Z}_{p,w} := \mathrm{alg}\left(aI, W^0(b) :\ a \in SO^\diamond,\ b \in SO^\diamond_{p,w}\right) \subset \mathcal{B}_{p,w},$$

which is generated by all the operators $aW^0(b) \in \mathcal{B}_{p,w}$ with $a \in SO^\diamond$ and $b \in SO^\diamond_{p,w}$. Then $\mathcal{K}_{p,w} \subset \mathcal{Z}_{p,w} \subset \mathfrak{A}_{p,w} \subset \mathcal{B}_{p,w}$.

Consider the quotient Banach algebra $\mathfrak{A}^\pi_{p,w} := \mathfrak{A}_{p,w}/\mathcal{K}_{p,w}$. Then $\mathcal{Z}^\pi_{p,w} := \mathcal{Z}_{p,w}/\mathcal{K}_{p,w}$ is a central subalgebra of $\mathfrak{A}^\pi_{p,w}$.

Take the following commutative Banach subalgebras of $\mathcal{Z}^\pi_{p,w}$:

$$\mathcal{A}^\pi_1 := \left\{(aI)^\pi : a \in SO^\diamond\right\},\quad \mathcal{A}^\pi_2 := \left\{(W^0(b))^\pi : b \in SO^\diamond_{p,w}\right\},$$

where $A^\pi := A + \mathcal{K}_{p,w}$ for all $A \in \mathcal{B}_{p,w}$, and their maximal ideals

$$\begin{aligned} \mathcal{I}^\pi_{1,\xi} &:= \left\{(aI)^\pi :\ a \in SO^\diamond,\ \xi(a) = 0\right\} \subset \mathcal{A}^\pi_1 \quad \text{and} \\ \mathcal{I}^\pi_{2,\eta} &:= \left\{(W^0(b))^\pi :\ b \in SO^\diamond_{p,w},\ \eta(b) = 0\right\} \subset \mathcal{A}^\pi_2. \end{aligned} \tag{1}$$

Lemma

If $1 < p < \infty$, $w \in A_p(\mathbf{R})$, *then* $\mathcal{A}^\pi_1 \cap \mathcal{A}^\pi_2 = \{cI^\pi : c \in \mathbf{C}\}$.

The maximal ideal space of $\mathcal{Z}^{\pi}_{p,w}$

Theorem

If $p \in (1,\infty)$ and $w \in A_p(\mathbf{R})$, then the maximal ideal space $M(\mathcal{Z}^{\pi}_{p,w})$ of the Banach algebra $\mathcal{Z}^{\pi}_{p,w}$ is homeomorphic to the set

$$\Omega := \Big(\bigcup_{\lambda\in\mathbf{R}} M_\lambda(SO^\diamond) \times M_\infty(SO^\diamond) \Big) \cup \Big(M_\infty(SO^\diamond) \times \bigcup_{\tau\in\mathbf{R}} M_\tau(SO^\diamond) \Big) \cup \big(M_\infty(SO^\diamond) \times M_\infty(SO^\diamond) \big)$$

equipped with topology induced by the product topology of $M(SO^\diamond) \times M(SO^\diamond)$, and the Gelfand transform $\Gamma_{\mathcal{Z}} : \mathcal{Z}^{\pi}_{p,w} \to C(\Omega)$, $A^\pi \mapsto \mathcal{A}(\cdot,\cdot)$ is defined on the generators $A^\pi = [aW^0(b)]^\pi$ $(a \in SO^\diamond,\ b \in SO^\diamond_{p,w})$ of the algebra $\mathcal{Z}^{\pi}_{p,w}$ by $\mathcal{A}(\xi,\eta) = a(\xi)b(\eta)$ for all $(\xi,\eta) \in \Omega$. An operator $A \in \mathcal{Z}_{p,w}$ is Fredholm on the space $L^p(\mathbf{R}, w)$ if and only if the Gelfand transform of the coset A^π is invertible, that is, if $\mathcal{A}(\xi,\eta) \neq 0$ for all $(\xi,\eta) \in \Omega$.

Scheme of the proof

If J is a maximal ideal of $\mathcal{Z}^{\pi}_{p,w}$, then $J \cap \mathcal{A}^{\pi}_1$ and $J \cap \mathcal{A}^{\pi}_2$ are maximal ideals of the algebras $\mathcal{A}^{\pi}_1$ and $\mathcal{A}^{\pi}_2$. Hence, for every $(\xi, \eta) \in M(SO^{\diamond}) \times M(SO^{\diamond})$ there is the closed two-sided (not necessarily maximal) ideal $\mathcal{I}^{\pi}_{\xi,\eta} \subset \mathcal{Z}^{\pi}_{p,w}$ generated by ideals (1).

If $(\xi, \eta) \in \bigcup_{t \in \mathbf{R}} M_t(SO^{\diamond}) \times \bigcup_{t \in \mathbf{R}} M_t(SO^{\diamond})$, then $\mathcal{I}^{\pi}_{\xi,\eta} = \mathcal{Z}^{\pi}_{p,w}$, and hence the ideal $\mathcal{I}^{\pi}_{\xi,\eta}$ is not proper.

Let us show that $\mathcal{I}^{\pi}_{\xi,\eta}$ are proper ideals of the algebra $\mathcal{Z}^{\pi}_{p,w}$ for all $(\xi, \eta) \in \Omega$. On the contrary, suppose that $\mathcal{I}^{\pi}_{\xi,\eta} = \mathcal{Z}^{\pi}_{p,w}$. Then $\mathcal{A}^{\pi}_1 \cap \mathcal{I}^{\pi}_{\xi,\eta} = \mathcal{A}^{\pi}_1 \cap \mathcal{Z}^{\pi}_{p,w}$. Since $b(\eta) = 0$ for each $[W^0(b)]^{\pi} \in \mathcal{I}^{\pi}_{2,\eta}$ by (1), we infer from the equality $\mathcal{A}^{\pi}_1 \cap \mathcal{A}^{\pi}_2 = \{cI^{\pi} : c \in \mathbf{C}\}$ that $\mathcal{A}^{\pi}_1 \cap \mathcal{I}^{\pi}_{2,\eta} = \{0^{\pi}\}$, where $0^{\pi} = \mathcal{K}_{p,w}$. Hence $\mathcal{A}^{\pi}_1 \cap \mathcal{I}^{\pi}_{\xi,\eta} = \mathcal{I}^{\pi}_{1,\xi}$.

On the other hand, $\mathcal{A}^{\pi}_1 \cap \mathcal{Z}^{\pi}_{p,w} = \mathcal{A}^{\pi}_1$, whence $\mathcal{I}^{\pi}_{1,\xi} = \mathcal{A}^{\pi}_1$, which is impossible because $\mathcal{I}^{\pi}_{1,\xi}$ is a maximal ideal of the algebra $\mathcal{A}^{\pi}_1$.

One can prove that the proper ideal $\mathcal{I}^{\pi}_{\xi,\eta}$ is maximal.

The algebras $\Lambda^\pi_{\xi,\eta}$ and $\mathcal{A}^\pi_{\xi,\eta}$

Let Λ^π denote the Banach algebra of all cosets $B^\pi \in \mathcal{B}^\pi_{p,w}$ that commute with all cosets $A^\pi \in \mathcal{Z}^\pi_{p,w}$. For any $(\xi,\eta) \in \Omega$, let $J^\pi_{\xi,\eta}$ be the smallest closed two-sided ideal of Λ^π generated by $\mathcal{I}^\pi_{\xi,\eta}$. Let $\mathcal{A}^\pi_{\xi,\eta}$ be the minimal closed subalgebra of the Banach algebra $\Lambda^\pi_{\xi,\eta} := \Lambda^\pi / J^\pi_{\xi,\eta}$ that contains the cosets $A^\pi_{\xi,\eta} := A^\pi + J^\pi_{\xi,\eta}$ for all $A \in \mathfrak{A}_{p,w}$.
For $c \in PSO^\diamond_{p,w}$ and $\xi \in M(SO^\diamond)$, we put

$$c(\xi^-) := c(\xi,0) \ \text{ and } \ c(\xi^+) := c(\xi,1), \tag{2}$$

where $c(\xi,\mu) = (\xi,\mu)c$ for $(\xi,\mu) \in M(SO^\diamond) \times \{0,1\}$.

For $t \in \mathbf{R}$, χ^-_t and χ^+_t denote the characteristic functions of $(-\infty,t)$ and $(t,+\infty)$, respectively, $\chi_\pm := \chi^\pm_0$, and for every $p \in (1,\infty)$, $w \in A_p(\mathbf{R})$, $\zeta \in M_\infty(SO)$, and $c \in PSO^\diamond_{p,w}$, the function $\widetilde{c}_\zeta \in C_{p,w}(\overline{\mathbf{R}})$ is given by $\widetilde{c}_\zeta := c(\zeta^+)u_- + c(\zeta^-)u_+$, where the values $c(\zeta^\pm)$ are defined in (2) and the functions $u_\pm \in C_{p,w}(\overline{\mathbf{R}})$ are given by $u_\pm(x) = (1 \pm \tanh x)/2$ for $x \in \mathbf{R}$.

Local representatives

Theorem

For every $(\xi,\eta)\in\Omega$ *and every* $t\in\mathbf{R}$*, the mapping* $\delta_{\xi,\eta}: A\mapsto A^{\pi}_{\xi,\eta}$ *defined on the generators* aI $(a\in PSO^{\diamond})$ *and* $W^0(b)$ $(b\in PSO^{\diamond}_{p,w})$ *of the Banach algebra* $\mathfrak{A}_{p,w}$ *by*

$$\delta_{\xi,\eta}(aI) := \begin{cases} [(a(\xi^-)\chi_t^- + a(\xi^+)\chi_t^+)I]^{\pi}_{\xi,\eta}, & (\xi,\eta)\in M_t(SO^{\diamond})\times M_\infty(SO^{\diamond}),\\ [(a(\xi^+)\chi_- + a(\xi^-)\chi_+)I]^{\pi}_{\xi,\eta}, & (\xi,\eta)\in M_\infty(SO^{\diamond})\times M_t(SO^{\diamond}),\\ [\widetilde{a}_\xi I]^{\pi}_{\xi,\eta}, & (\xi,\eta)\in M_\infty(SO^{\diamond})\times M_\infty(SO^{\diamond}); \end{cases}$$

$$\delta_{\xi,\eta}(W^0(b)) := \begin{cases} [W^0(b(\eta^+)\chi_- + b(\eta^-)\chi_+)]^{\pi}_{\xi,\eta}, & (\xi,\eta)\in M_t(SO^{\diamond})\times M_\infty(SO^{\diamond}),\\ [W^0(b(\eta^-)\chi_t^- + b(\eta^+)\chi_t^+)]^{\pi}_{\xi,\eta}, & (\xi,\eta)\in M_\infty(SO^{\diamond})\times M_t(SO^{\diamond}),\\ [W^0(\widetilde{b}_\eta)]^{\pi}_{\xi,\eta}, & (\xi,\eta)\in M_\infty(SO^{\diamond})\times M_\infty(SO^{\diamond}) \end{cases}$$

extends to a Banach algebra homomorphism $\delta_{\xi,\eta}:\mathfrak{A}_{p,w}\to\mathcal{A}^{\pi}_{\xi,\eta}$.

Corollary of the Allan-Douglas local principle

Moreover, for all $A \in \mathfrak{A}_{p,w}$,

$$\sup_{(\xi,\eta)\in\Omega} \|\delta_{\xi,\eta}(A)\|_{\mathcal{A}^\pi_{\xi,\eta}} \le \|A^\pi\| := \inf_{K\in\mathcal{K}_{p,w}} \|A+K\|.$$

This theorem and the Allan-Douglas local principle imply

Corollary

An operator $A \in \mathfrak{A}_{p,w}$ is Fredholm on the space $L^p(\mathbf{R}, w)$ if and only if the cosets $A^\pi_{\xi,\eta} = \delta_{\xi,\eta}(A) \in \mathcal{A}^\pi_{\xi,\eta}$ are invertible in the quotient algebras $\Lambda^\pi_{\xi,\eta}$ for all $(\xi,\eta) \in \Omega$.

The following result shows that for every $(\xi,\eta) \in \Omega_0$, where

$$\Omega_0 := \Big(\bigcup_{t\in\mathbf{R}} M_t(SO^\diamond) \times M_\infty(SO^\diamond)\Big) \cup \Big(M_\infty(SO^\diamond) \times \bigcup_{t\in\mathbf{R}} M_t(SO^\diamond)\Big),$$

the Banach algebra $\mathcal{A}^\pi_{\xi,\eta}$ satisfies conditions of the two idempotents theorem, while $\mathcal{A}^\pi_{\xi,\eta}$ for every $(\xi,\eta) \in M_\infty(SO^\diamond) \times M_\infty(SO^\diamond)$ is a commutative Banach algebra isomorphic to $C(\mathcal{X}, \mathbf{C})$, where $\mathcal{X} := \{(\pm\infty, \pm\infty)\}$.

Structure of local algebras $\mathcal{A}^{\pi}_{p,w}$

Lemma

The local algebras $\mathcal{A}^{\pi}_{\xi,\eta}$ generated by all the cosets $[aW^0(b)]^{\pi}_{\xi,\eta} = [aW^0(b)]^{\pi} + J^{\pi}_{\xi,\eta}$, where $a \in PSO^{\diamond}$ and $b \in PSO^{\diamond}_{p,w}$, have the following structure:

(i) *if $(\xi,\eta) \in M_t(SO^{\diamond}) \times M_{\infty}(SO^{\diamond})$ and $t \in \mathbf{R}$, then $\mathcal{A}^{\pi}_{\xi,\eta}$ is generated by the unit $I^{\pi}_{\xi,\eta}$ and the two idempotents*

$$P = [\chi_t^+ I]^{\pi}_{\xi,\eta}, \quad Q = [W^0(\chi_0^-)]^{\pi}_{\xi,\eta};$$

(ii) *if $(\xi,\eta) \in M_{\infty}(SO^{\diamond}) \times M_t(SO^{\diamond})$ and $t \in \mathbf{R}$, then $\mathcal{A}^{\pi}_{\xi,\eta}$ is generated by the unit $I^{\pi}_{\xi,\eta}$ and the two idempotents*

$$P = [\chi_0^- I]^{\pi}_{\xi,\eta}, \quad Q = [W^0(\chi_t^+)]^{\pi}_{\xi,\eta};$$

(iii) *if $(\xi,\eta) \in M_{\infty}(SO^{\diamond}) \times M_{\infty}(SO^{\diamond})$, then $\mathcal{A}^{\pi}_{\xi,\eta}$ is generated by the unit $I^{\pi}_{\xi,\eta}$ and the two mutually commuting idempotents*

$$P = [u_- I]^{\pi}_{\xi,\eta}, \quad Q = [W^0(u_-)]^{\pi}_{\xi,\eta}.$$

where the function $u_-(x) = [1 - \tanh x]/2$.

Indices $\nu_t^{\pm}(p, w)$

Let $1 < p < \infty$, $w \in A_p(\mathbf{R})$, and $I(t, \varepsilon) = (t - \varepsilon, t + \varepsilon)$ where $t \in \mathbf{R}$, $\varepsilon > 0$. By [Böttcher/Yu. K.], with a weight $w \in A_p(\mathbf{R})$ and every point $t \in \mathbf{R}$ one can associate a submultiplicative function $V_t^0 w : (0, \infty) \to (0, \infty)$ and its indices,

$$(V_t^0 w)(x) := \limsup_{R \to 0} \exp\left(\frac{1}{2xR} \int_{I(t,xR)} \log w(\tau)\, d\tau - \frac{1}{2R} \int_{I(t,R)} \log w(\tau)\, d\tau\right),$$

$$\alpha_t(w) := \lim_{x \to 0} \frac{\log(V_t^0 w)(x)}{\log x}, \quad \beta_t(w) := \lim_{x \to \infty} \frac{\log(V_t^0 w)(x)}{\log x}.$$

As is known, $-1/p < \alpha_t(w) \le \beta_t(w) < 1/q$. Therefore,

$$0 < \nu_t^-(p, w) := 1/p + \alpha_t(w) \le \nu_t^+(p, w) := 1/p + \beta_t(w) < 1$$

for all $t \in \mathbf{R}$ and also for $t = \infty$. Consider the horn

$$\mathcal{L}_{p,w,t} := \bigcup_{\nu \in [\nu_t^-(p,w), \nu_t^+(p,w)]} \left\{ (1 + \coth[\pi(x + i\nu)])/2 : x \in \overline{\mathbf{R}} \right\}.$$

Local spectra

Let $w = e^v$ be a slowly oscillating weight in $A_p(\mathbf{R})$, i.e., for every $\lambda \in \dot{\mathbf{R}}$ the function

$$\sigma_\lambda(x) := \begin{cases} (x-\lambda)v'(x) & \text{if } \lambda \in \mathbf{R}, \\ xv'(x) & \text{if } \lambda = \infty, \end{cases}$$

is in $SO_\lambda(u_\lambda)$, where u_λ is a neighborhood of λ on $\dot{\mathbf{R}}$. For all $\lambda \in \dot{\mathbf{R}}$ and all $\xi \in M_\lambda(SO^\diamond)$, we define $\delta_\xi := \xi(\sigma_\lambda)$.
For any $t \in \dot{\mathbf{R}}$ and any $\xi \in M_t(SO^\diamond)$, we define the circular arc

$$\widetilde{\mathcal{L}}_{p,w,\nu(\xi)} := \left\{ \left(1 + \coth[\pi(x + i\nu(\xi))]\right)/2 : \ x \in \overline{\mathbf{R}} \right\} \subset \mathcal{L}_{p,w,t}$$
$$\nu(\xi) := 1/p + \delta_\xi \in (0,1).$$

Applying now the two idempotents theorem, the theory of Mellin pseudodifferential operators with slowly oscillating $V(\mathbf{R})$-valued symbols on $\mathbf{R}_+$ and technique of limit operators to studying local algebras $\mathcal{A}_{\xi,\eta}$ for all $(\xi,\eta) \in \Omega$ and calculating their local spectra, we construct the following Fredholm symbol calculus.

Commutative Banach algebra $\mathcal{Y}^{\pi}_{\lambda,\tau}$

Fix $(\lambda,\tau)\in\widehat{\Omega}_0 := \big(\mathbf{R}\times\{\infty\}\big)\cup\big(\{\infty\}\times\mathbf{R}\big)$. Consider the commutative Banach algebra $\mathcal{Y}^{\pi}_{\lambda,\tau}$ generated by the cosets $[aI]^{\pi}$ ($a\in SO^{\diamond}$), $[W^0(b)]^{\pi}$ ($b\in SO^{\diamond}_{p,w}$) and $[\widehat{X}_{\lambda,\tau}]^{\pi}$, where

$$\widehat{X}_{\lambda,\tau} := \begin{cases} I-(\chi^{+}_{\lambda}I-W^0(\chi^{-}_{0}))^2 & \text{if } (\lambda,\tau)\in\mathbf{R}\times\{\infty\},\\ I-(\chi^{-}_{0}I-W^0(\chi^{+}_{\tau}))^2 & \text{if } (\lambda,\tau)\in\{\infty\}\times\mathbf{R}.\end{cases}$$

For every $(\xi,\eta,\mu)\in M_{\lambda}(SO^{\diamond})\times M_{\tau}(SO^{\diamond})\times \mathrm{sp}_{ess}\widehat{X}_{\lambda,\tau}$, we introduce the closed two-sided ideal $\mathcal{I}^{\pi}_{\xi,\eta,\mu}$ of the commutative Banach algebra $\mathcal{Y}^{\pi}_{\lambda,\tau}$ generated by the maximal ideals

$$\mathcal{I}^{\pi}_{1,\xi} := \big\{[aI]^{\pi} :\ a\in SO^{\diamond},\ a(\xi)=0\big\},$$
$$\mathcal{I}^{\pi}_{2,\eta} := \big\{[W^0(b)]^{\pi} :\ b\in SO^{\diamond}_{p,w},\ b(\eta)=0\big\},$$
$$\mathcal{I}^{\pi}_{3,\mu} \quad \text{that contains} \quad [\widehat{X}_{\lambda,\tau}-\mu I]^{\pi}$$

of the commutative Banach algebras $\mathcal{A}^{\pi}_1 = \big\{[aI]^{\pi} : a\in SO^{\diamond}\big\}$,

$$\mathcal{A}^{\pi}_2 = \big\{[W^0(b)]^{\pi} : b\in SO^{\diamond}_{p,w}\big\},\quad \mathcal{D}^{\pi}_{\lambda,\tau} := \mathrm{alg}\,([\widehat{X}_{\lambda,\tau}]^{\pi}).$$

Spectrum of the coset $X^\pi_{\xi,\eta}$

For all $(\xi, \eta) \in \Omega_0$, let us determine the spectra of elements $X^\pi_{\xi,\eta} := [\widehat{X}_{\lambda,\tau}]^\pi + J^\pi_{\xi,\eta}$, crucial in the two idempotents theorem.

Theorem

If $(\xi, \eta) \in \Omega_0$, *then* $\{0, 1\} \subset \mathrm{sp}_{\Lambda^\pi_{\xi,\eta}} X^\pi_{\xi,\eta} \subset \widetilde{\mathcal{L}}_{p,w,\nu(\xi)}$.

Given $(\xi, \eta) \in M_\lambda(SO^\diamond) \times M_\tau(SO^\diamond)$, where $(\lambda, \tau) \in \widehat{\Omega}_0$, let

$$\mathfrak{M}_{\xi,\eta} := \{\mu \in \widetilde{\mathcal{L}}_{p,w,\nu(\xi)} : I^\pi \notin \mathcal{I}^\pi_{\xi,\eta,\mu}\}. \tag{3}$$

Theorem

If $(\xi, \eta) \in M_\lambda(SO^\diamond) \times M_\tau(SO^\diamond)$ *and* $(\lambda, \tau) \in \widehat{\Omega}_0$, *then*

$$\mathrm{sp}_{\Lambda^\pi_{\xi,\eta}} X^\pi_{\xi,\eta} = \mathrm{sp}_{\mathcal{A}^\pi_{\xi,\eta}} X^\pi_{\xi,\eta} = \mathfrak{M}_{\xi,\eta}.$$

Lemma

The Banach algebras $\mathcal{A}^\pi_1$, $\mathcal{A}^\pi_2$ *and* $\mathcal{D}^\pi_{\lambda,\tau}$ *possess the properties:*

$$\mathcal{A}^\pi_1 \cap \mathcal{A}^\pi_2 = \{cI^\pi : c \in \mathbf{C}\}, \quad \mathcal{A}^\pi_1 \cap \mathcal{D}^\pi_{\lambda,\tau} = \{cI^\pi : c \in \mathbf{C}\}. \tag{4}$$

Identification of the local spectra

Theorem

$$\mathfrak{M}_{\xi,\eta} = \widetilde{\mathcal{L}}_{p,w,\nu(\xi)} \quad \textit{for every} \quad (\xi,\eta) \in \Omega_0. \tag{5}$$

Fix $(\xi,\eta) \in \Omega_0$ and $\mu \in \widetilde{\mathcal{L}}_{p,w,\nu(\xi)}$. To prove (5), it is sufficient to show in view of (3) that $\mathcal{I}^\pi_{\xi,\eta,\mu} \neq \mathcal{Y}^\pi_{\lambda,\tau}$. On the contrary, suppose that $\mathcal{I}^\pi_{\xi,\eta,\mu} = \mathcal{Y}^\pi_{\lambda,\tau}$. Then

$$\mathcal{A}^\pi_1 \cap \mathcal{I}^\pi_{\xi,\eta,\mu} = \mathcal{A}^\pi_1 \cap \mathcal{Y}^\pi_{\lambda,\tau}. \tag{6}$$

It immediately follows from (4) that

$$\mathcal{A}^\pi_1 \cap \mathcal{I}^\pi_{2,\eta} = \mathcal{A}^\pi_1 \cap \mathcal{I}^\pi_{3,\mu} = \mathcal{K}_{p,w} \;\Rightarrow\; \mathcal{A}^\pi_1 \cap \mathcal{I}^\pi_{\xi,\eta,\mu} = \mathcal{I}^\pi_{1,\xi}. \tag{7}$$

But $\mathcal{A}^\pi_1 \cap \mathcal{Y}^\pi_{\lambda,\tau} = \mathcal{A}^\pi_1$. Combining this with (6) and (7), we obtain the equality $\mathcal{I}^\pi_{1,\xi} = \mathcal{A}^\pi_1$, which is impossible because $\mathcal{I}^\pi_{1,\xi}$ is a maximal ideal of the commutative Banach algebra $\mathcal{A}^\pi_1$. Hence, $I^\pi \notin \mathcal{I}^\pi_{\xi,\eta,\mu}$ for all $\mu \in \widetilde{\mathcal{L}}_{p,w,\nu(\xi)}$, which gives (5).

Symbol calculus for the Banach algebra $\mathfrak{A}_{p,w}$

Let $1 < p < \infty$, let $w = e^v \in A_p(\mathbf{R})$ be a slowly oscillating weight, and let

$$\widetilde{\Omega} := \Big(\bigcup_{(\xi,\eta)\in\Omega_0} \{(\xi,\eta)\} \times \widetilde{\mathcal{L}}_{p,w,\nu(\xi)} \Big) \cup \Big(M_\infty(SO^\diamond) \times M_\infty(SO^\diamond) \times \{0,1\} \Big),$$

$$\Omega_0 := \Big(\bigcup_{\lambda\in\mathbf{R}} M_\lambda(SO^\diamond) \times M_\infty(SO^\diamond) \Big) \cup \Big(M_\infty(SO^\diamond) \times \bigcup_{\tau\in\mathbf{R}} M_\tau(SO^\diamond) \Big).$$

For each $(\xi,\eta,\mu) \in \widetilde{\Omega}$, we consider the mapping

$$\Psi_{\xi,\eta,\mu} : \; \{aI : a \in PSO^\diamond\} \cup \{W^0(b) : b \in PSO^\diamond_{p,w}\} \to \mathbf{C}^{2\times 2}$$

given by the following formulas:

$$\Psi_{\xi,\eta,\mu}(aI) := \operatorname{diag}\{a(\xi^+), a(\xi^-)\},$$

$$\Psi_{\xi,\eta,\mu}(W^0(b)) := \begin{bmatrix} b(\eta^+)\mu + b(\eta^-)(1-\mu) & [b(\eta^+) - b(\eta^-)]\varrho(\mu) \\ [b(\eta^+) - b(\eta^-)]\varrho(\mu) & b(\eta^+)(1-\mu) + b(\eta^-)\mu \end{bmatrix},$$

where $\varrho(\mu)$ is any fixed value of $\sqrt{\mu(1-\mu)}$.

Fredholmness

Theorem

The mappings $\Psi_{\xi,\eta,\mu}$ $\big((\xi,\eta,\mu)\in\widetilde{\Omega}\big)$*, given on the generators of the Banach algebra* $\mathfrak{A}_{p,w}$ *by the formulas above, extend to Banach algebra homomorphisms* $\Psi_{\xi,\eta,\mu}:\mathfrak{A}_{p,w}\to\mathbf{C}^{2\times 2}$*.*

Hence, the symbol mapping $\Psi:\ \mathfrak{A}_{p,w}\to B(\widetilde{\Omega},\mathbf{C}^{2\times 2}),\ \ A\mapsto\mathcal{A}$ into the Banach algebra $B(\widetilde{\Omega},\mathbf{C}^{2\times 2})$ of bounded matrix functions

$$\mathcal{A}:\ \widetilde{\Omega}\to\mathbf{C}^{2\times 2},\quad (\xi,\eta,\mu)\mapsto\mathcal{A}(\xi,\eta,\mu):=\Psi_{\xi,\eta,\mu}(A)$$

called *symbols* of operators $A\in\mathfrak{A}_{p,w}$ is a Banach algebra homomorphism whose kernel $\mathrm{Ker}\,\Psi$ contains the ideal $\mathcal{K}_{p,w}$.

Theorem

If $1<p<\infty$ *and* $w\in A_p(\mathbf{R})$ *is a slowly oscillating weight, then the Banach algebra* $\mathfrak{A}^{\pi}_{p,w}$ *is inverse closed in the Calkin algebra* $\mathcal{B}^{\pi}_{p,w}$*. An operator* $A\in\mathfrak{A}_{p,w}$ *is Fredholm on the space* $L^p(\mathbf{R},w)$ *if and only if* $\det\mathcal{A}(\xi,\eta,\mu)\neq 0$ *for all* $(\xi,\eta,\mu)\in\widetilde{\Omega}$*.*