Relationship between the effective thermal properties of linear and nonlinear doubly periodic composites

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Based on the work

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The study of composites is a subject with a long history, which has attracted the interest of some of the greatest scientists. For example, Poisson (1826) constructed a theory of induced magnetism in which the body was assumed to be composed of conducting spheres embedded in a nonconducting material. Faraday (1839) proposed a model for dielectric materials that consisted of metallic globules separated by insulating material. Maxwell (1873) solved for the conductivity of a dilute suspension of conducting spheres in a conducting matrix. Rayleigh (1892) found a system of linear equations which, when solved, would give the effective conductivity of undilute square arrays of cylinders or cubic lattices of spheres. Einstein (1905) calculated the effective shear viscosity of a suspension of rigid spheres in a fluid.

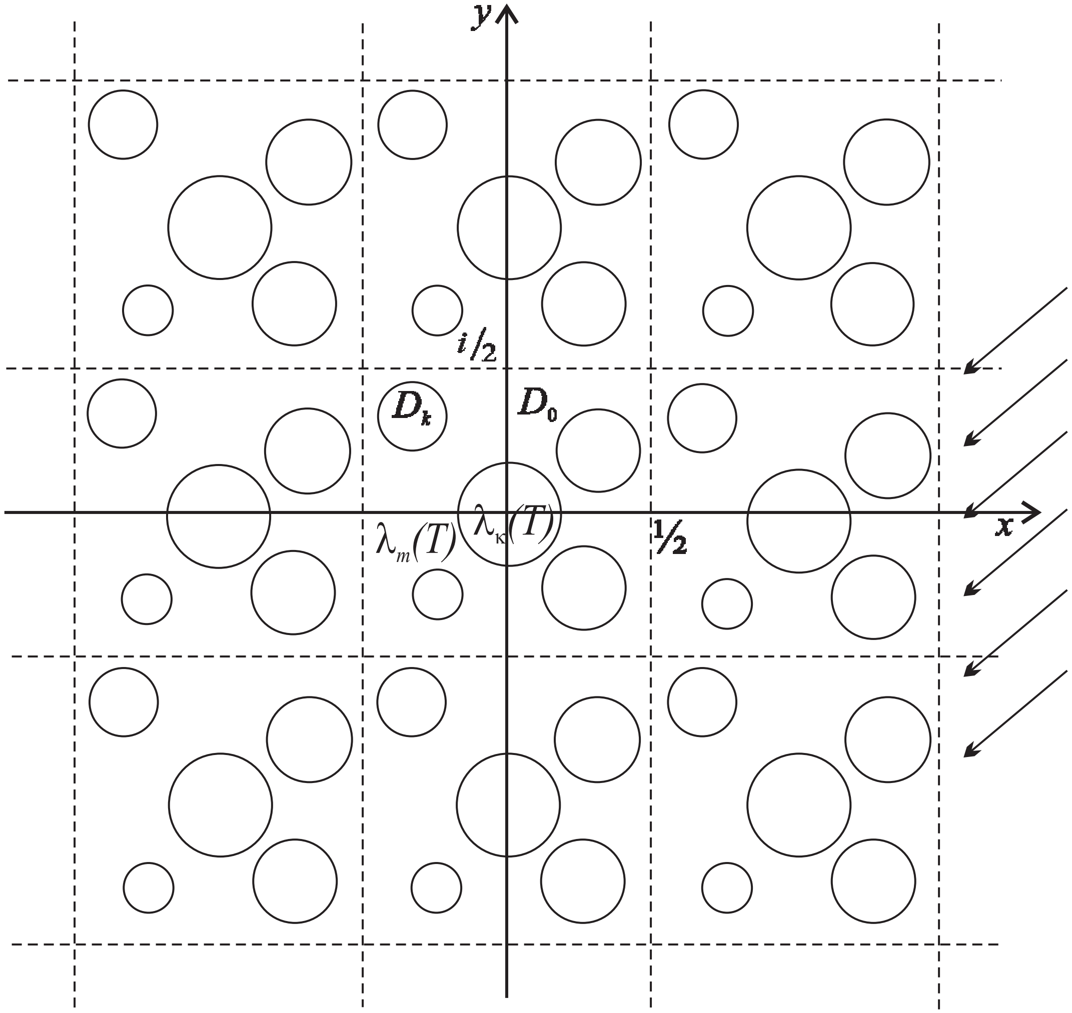
The field of composite materials is enormous. There are many avenues of research to explore. We are mainly interested in the relation between the microstructure of composites and the effective moduli that govern their behavior. The respective tasks are highly interdisciplinary bringing together specialists from mathematics, physics and engineering working with composite materials.

There exist a lot of natural and man-made composites.

Examples of natural composites:
Common metals are composites. Bone is a porous composite. Clouds, fog, mist, and rain are composites of air and water. High-altitude clouds are composites of air and ice crystals. Ceramics are composites.

A composite material (also called a composition material or shortened to composite) is a material made from two or more constituent materials with significantly different physical or chemical properties that, when combined, produce a material with characteristics different from the individual components.

Basically, composites are materials that have inhomogeneities on length scales that are much larger than the atomic scale (which allows us to use the equations of classical physics at the length scales of the inhomogeneities) but which are essentially homogeneous at macroscopic length scales, or least at some intermediate length scales.



The doubly periodic composite with nonconstant conductivities of the matrix and inclusions. The representative cell $Q_{(0,0)}$ is a unit square. Non-overlapping inclusions are disks. They are located inside the cell $Q_{(0,0)}$ and periodically repeated in all cells $Q_{(m_1,m_2)}$.

On a macroscopic length scale the composite behaves exactly like a homogeneous medium and the goal is to find its effective properties, such as, the effective conductivity based on the conductivities of the matrix and inclusions.

Statement of the problem

$$z = x + \imath y, \ \ \mathbb{C} \cong \mathbb{R}^2, \ \ \imath^2 = -1.$$

The representative cell $Q_{(0,0)}$ is a unit square. Other cells is denoted by

$$Q_{(m_1,m_2)} = Q_{(0,0)} + m_1 + \imath m_2 := \left\{z \in \mathbb{C} : z - m_1 - \imath m_2 \in Q_{(0,0)}\right\},$$

where $m_1, m_2 \in \mathbb{Z}$.

Non-overlapping inclusions are disks

$$D_k := \{z \in \mathbb{C} : |z - a_k| < r_k\}, \ (k = 1, 2, \ldots, N),$$

with boundaries

$$\partial D_k := \{z \in \mathbb{C} : |z - a_k| = r_k\}$$

are located inside the cell $Q_{(0,0)}$ and periodically repeated in all cells $Q_{(m_1,m_2)}$. The connected domain D_0 obtained by removing of the inclusions from the cell $Q_{(0,0)}$:

$$D_0 := Q_{(0,0)} \setminus \left(\bigcup_{k=1}^{N} D_k \cup \partial D_k \right).$$

$$D_{matrix} = \bigcup_{m_1, m_2} ((D_0 \cup \partial Q_{(0,0)}) + m_1 + \imath m_2)$$

$$D_{inc} = \bigcup_{m_1, m_2} \bigcup_{k=1}^{N} (D_k + m_1 + \imath m_2)$$

with thermal-sensitive conductivities

$$0 < \lambda(T) < +\infty, \qquad 0 < \lambda_k(T) < +\infty, \quad k = 1, \ldots, N.$$

We search for steady-state distribution of temperature and heat flux within such composite:

$$\begin{aligned} &\nabla(\lambda(T)\nabla T) = 0, \ (x, y) \in D_{matrix}, \\ &\nabla(\lambda_k(T)\nabla T) = 0, \ (x, y) \in D_{inc}. \end{aligned} \tag{1}$$

Ideal-contact conditions on the boundaries of the matrix and inclusions:

$$\begin{aligned} &T(t) = T_k(t), \quad t \in \bigcup_{m_1, m_2 \in \mathbb{Z}} (\partial D_k + m_1 + \imath m_2), \\ &\lambda(T(t))\frac{\partial T(t)}{\partial n} = \lambda_k(T_k(t))\frac{\partial T_k(t)}{\partial n}, \quad t \in \bigcup_{m_1, m_2 \in \mathbb{Z}} (\partial D_k + m_1 + \imath m_2). \end{aligned} \tag{2}$$

According to the formulation, the flux and the temperature are continuous functions in the entire structure.

Interface conditions on the horizontal boundaries of each cell when $-1/2 < x < 1/2$ can be written in the following form:

$$\begin{aligned} \lambda(T)T_y|_{\partial Q^{(top)}_{(m_1,m_2)}} &= -A\sin\theta + q_1^+(x+m_1, m_1, m_2), \\ \lambda(T)T_y|_{\partial Q^{(bottom)}_{(m_1,m_2)}} &= -A\sin\theta + q_1^-(x+m_1, m_1, m_2). \end{aligned} \tag{3}$$

On the vertical boundaries when $-1/2 < y < 1/2$, we write

$$\begin{aligned} \lambda(T)T_x|_{\partial Q^{(left)}_{(m_1,m_2)}} &= -A\cos\theta + q_2^-(y+m_2, m_1, m_2), \\ \lambda(T)T_x|_{\partial Q^{(right)}_{(m_1,m_2)}} &= -A\cos\theta + q_2^+(y+m_2, m_1, m_2), \end{aligned} \tag{4}$$

where the unknown functions $q_j^{\pm}(\cdot, m_1, m_2)$

$$\int_{-1/2}^{1/2} q_j^{\pm}(\xi + m_j, m_1, m_2)d\xi = 0.$$

The average flux vector of intensity A is directed at an angle θ to axis Ox which does not coincide, in general, with the orientation of the periodic cell:

Linear composite

The problem for the linear composite can be completely solved by various methods. Here, we present only some results important for the nonlinear composite considered by us.

Theorem: For any given data θ and A the linear boundary value problem (1)-(4) (i.e., $\lambda(T) = \lambda = const$, $\lambda_k(T) = \lambda_k = const$) with ideal (perfect) contact conditions has a unique (modulo real constants) real analytic solution.

Note that an additional condition on the temperature T such as

$$T(x_0, y_0) = t_0, \quad (x_0, y_0) \in D_{matrix} \cup D_{inc}, \tag{5}$$

where t_0 is a given value of the temperature at any point (x_0, y_0), will give us a unique solvability result. Specifically, the following corollary holds true.

Corollary: The boundary value problem (1)-(5) has a unique real analytic solution.

Note that the calculation of the effective conductivity tensor Λ defined by the formula

$$\langle \lambda_l \nabla T \rangle = \Lambda \langle \nabla T \rangle, \tag{6}$$

where

$$\lambda_l = \lambda_l(z) = \begin{cases} \lambda, \ z \in D_{matrix}, \\ \\ \lambda_k, \ z \in D_{inc}, \end{cases}$$

does not depend on the chosen cell. Moreover, since the constants disappear in the expression ∇T, the effective properties of the linear composites are independent of the additional condition (5) and, crucially, of the applied flux A.

Nonlinear composite with proportional component conductivities

$$\frac{\lambda(T)}{\lambda_k(T)} = C_k$$

We use the Baiocchi (Kirchhoff) transformation and introduce continuous increasing monotonic functions $f : \mathbb{R} \to \mathbb{R}$, $f_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, N$,

$$f(T) = \int_0^T \lambda(\xi)\, d\xi, \quad f_k(T) = \int_0^T \lambda_k(\xi)\, d\xi,$$

$$u(x,y) = f(T(x,y)), \quad u_k(x,y) = f_k(T_k(x,y)).$$

$$\Delta u = 0, \quad (x,y) \in D_{matrix},$$

$$\Delta u_k = 0, \quad (x,y) \in \bigcup_{m_1, m_2 \in \mathbb{Z}} D_k + m_1 + \imath m_2.$$

$$u_y|_{\partial Q^{(top)}_{(m_1,m_2)}} = -A \sin\theta + q_1^+(x + m_1, m_1, m_2),$$

$$u_y|_{\partial Q^{(bottom)}_{(m_1,m_2)}} = -A \sin\theta + q_1^-(x + m_1, m_1, m_2),$$

$$u_x\big|_{\partial Q^{(left)}_{(m_1,m_2)}} = -A\cos\theta + q_2^-(y+m_2, m_1, m_2),$$

$$u_x\big|_{\partial Q^{(right)}_{(m_1,m_2)}} = -A\cos\theta + q_2^+(y+m_2, m_1, m_2).$$

$$u = F_k(u_k),$$

$$\frac{\partial u}{\partial n} = \frac{\partial u_k}{\partial n}, \quad (x,y) \in \bigcup_{m_1,m_2\in\mathbb{Z}} (\partial D_k + m_1 + \imath m_2),$$

where the functions $F_k(\xi) := f(f_k^{-1}(\xi))$ are defined for an arbitrary $\xi \in \mathbb{R}$.

Differentiating the function F_k, we get

$$F_k'(\xi) = \frac{f'(f_k^{-1}(\xi))}{f_k'(f_k^{-1}(\xi))} = \frac{\lambda(T_k)}{\lambda_k(T_k)},$$

$$\frac{\lambda(T)}{\lambda_k(T)} = C_k$$

D. Kapanadze, G. Mishuris, E. Pesetskaya: Exact solution of a nonlinear heat conduction problem in a doubly periodic 2D composite material, Archives of Mechanics, 67(2), 157–178, 2015

Theorem: The boundary value problem (1)–(4), (5) with proportional component conductivities has a unique real analytic solution.

In fact we have established a bijection between solutions of the linear and nonlinear boundary value problems via the Kirchhoff transformation which allows us to describe a solution of the boundary value problem (1)–(4) complemented by the condition (5), and to prove the effectiveness of the numerical algorithm for evaluation of the effective properties of the composite.

Let us denote by T_n a solution to our nonlinear boundary value problem, and

$$\lambda_n(T_n(z)) = \begin{cases} \lambda(T_n(z)), \ z \in D_{matrix}, \\ \\ \lambda_k(T_n(z)), \ z \in D_{inc}. \end{cases} \tag{7}$$

We define the effective conductivity tensor Λ_n of the representative cell of a nonlinear composite by the same formula (6):

$$\langle \lambda_n(T_n) \nabla T_n \rangle = \Lambda_n \langle \nabla T_n \rangle. \tag{8}$$

Effective conductivity tensor

It was shown that, in contrast to the linear composites, the effective conductivity tensor Λ_n varies from cell to cell and thus represents a function of the problem solution. The question arises: what would be a minimal set of variables that defines this function uniquely?

Hypothesis : $\Lambda_n(\langle T\rangle)$!

For a chosen example of the nonlinear composite and the averaged flux flowing through it, we have shown that the effective properties of the composite can be attributed to the average temperature:

$$\Lambda_n = \Lambda_n(\langle T_n\rangle). \tag{9}$$

$$\langle \lambda(T)\nabla T\rangle = \Lambda_n(\langle T\rangle)\langle \nabla T\rangle.$$

We have shown that the tensor of the effective properties, Λ_n, defined according to (7), may depend not only on the average temperature, $\langle T_n\rangle$, but also the flux intensity, A,

$$\Lambda_n = \Lambda_n(\langle T_n\rangle, |A|). \tag{10}$$

The natural question may then be asked of whether such a periodic structure can be represented as a composite material possessing average properties, or whether it is only the nonlinear periodic structure and the respective physical problems that should be considered in the original formulation, without reference to its effective properties.

Relationship between the effective conductivity tensors of nonlinear and respective linear models

Note that in some practical problems we may have sufficiently small $|\lambda_m(T_n(z)) - \lambda_m(\langle T_n(z)\rangle)|$ for all $z \in Q_{(0,0)}$. In this case we have

$$\Lambda_n \approx \lambda(\langle T_n(z)\rangle)\Lambda_l(C^{-1}, 1) \tag{11}$$

Here, $\Lambda_l(\lambda_k, \lambda)$ denotes the effective conductivity tensor for the respective linear problem and

$$\frac{\lambda(T)}{\lambda_k(T)} = C.$$

Theorem: Let the functions f defined as

$$f(T) = \int_0^T \lambda(\xi)\, d\xi,$$

and λ be Lipschitz continuous with Lipschitz constants C_f and C_λ, respectively. Then for any arbitrarily small value $\varepsilon > 0$, there exists $A_* > 0$, ($A_* = A_*(\varepsilon)$) such that for any intensity satisfying an equality $|A| < A_*$ the following estimate holds,

$$\left|(\Lambda_n)_{ij} - \lambda_m(\langle T_n(z)\rangle)\Lambda_l(C^{-1}, 1)_{ij}\right| < \varepsilon, \tag{12}$$

for all $i, j = 1, 2$.

Note, that $|A| \leq A_* = \varepsilon a_*$, where

$$a_* := \frac{\lambda_{min}^2}{2\lambda_{max}\Lambda_{max}C_\lambda C_f M}, \quad \Lambda_{max} := \max\{\lambda_{max}, \lambda_{k,max}\}$$

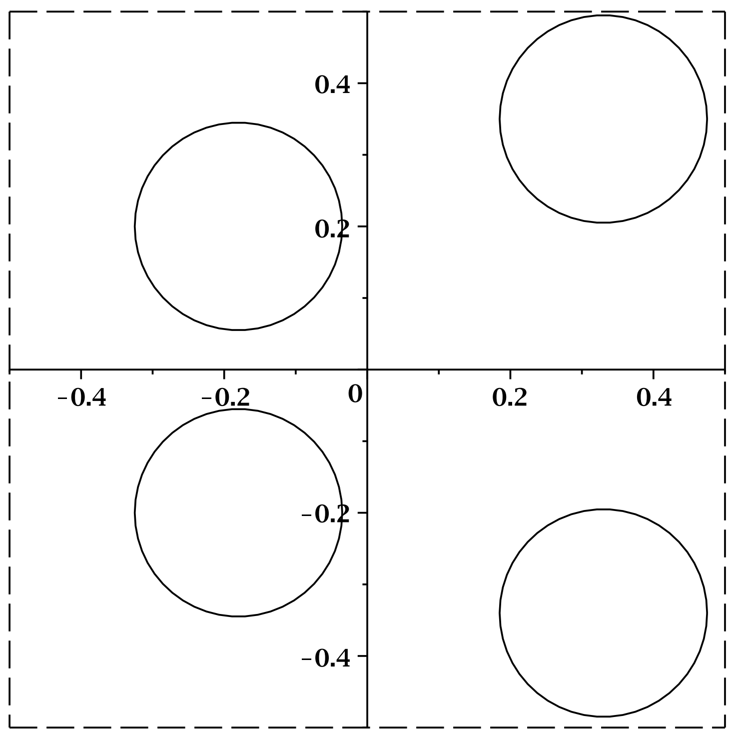
the constant M depends on the contrast parameter C as well as the specific distribution of the inclusions within the unit cell for the respective linear composite.

Numerical example

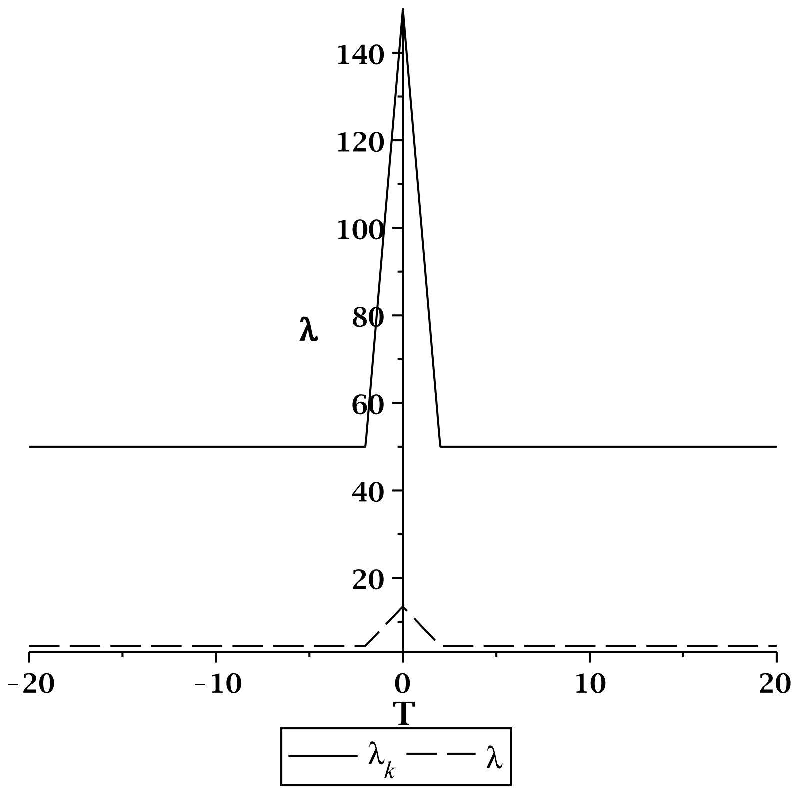
$$a_1 = -0.18 + 0.2\imath,\ a_2 = 0.33 - 0.34\imath,$$

$$a_3 = 0.33 + 0.35\imath,\ a_4 = -0.18 - 0.2\imath,$$

$$r_k = R = 0.145, \qquad \theta = 0,\ A = -1.$$



The conductivities of the matrix and inclusions:

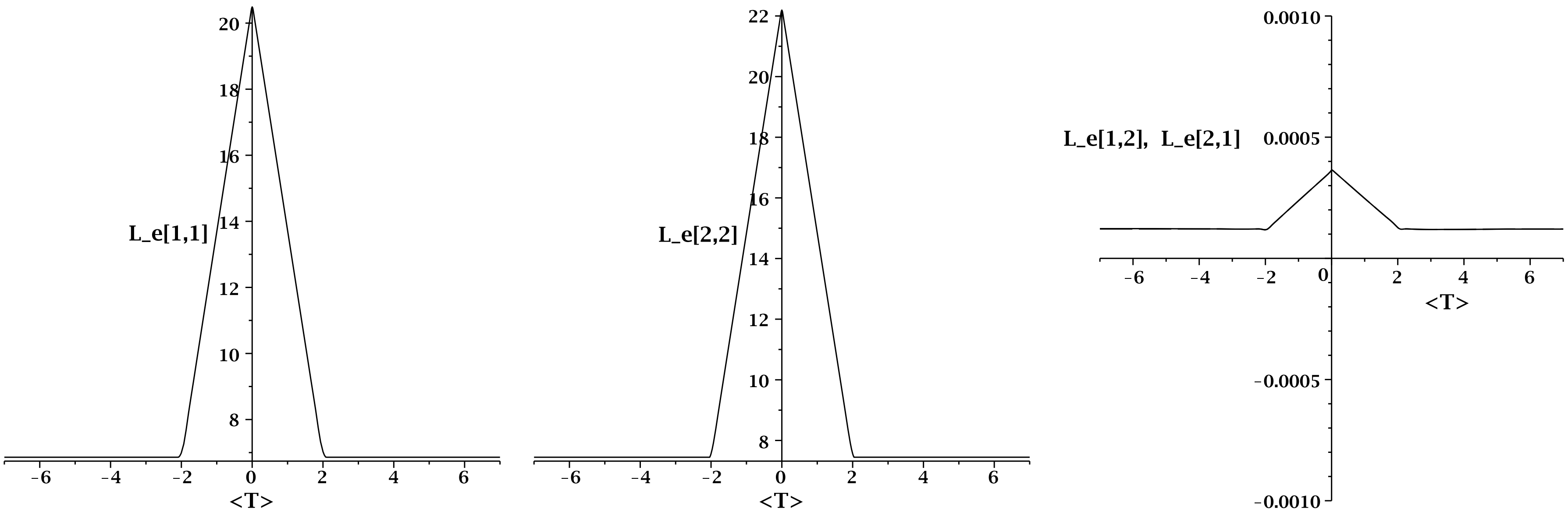


For computations, we use the algorithm described in

D. Kapanadze, G. Mishuris, E. Pesetskaya, Improved algorithm for analytical solution of the heat conduction problem in composites. Complex Variables and Elliptic Equations, 60(1), 1–23, 2015

where a solution found in a form of the Taylor series. For the chosen configuration, we take $M = 4$ first items in the Taylor series giving us a computational error less than 10^{-6}.

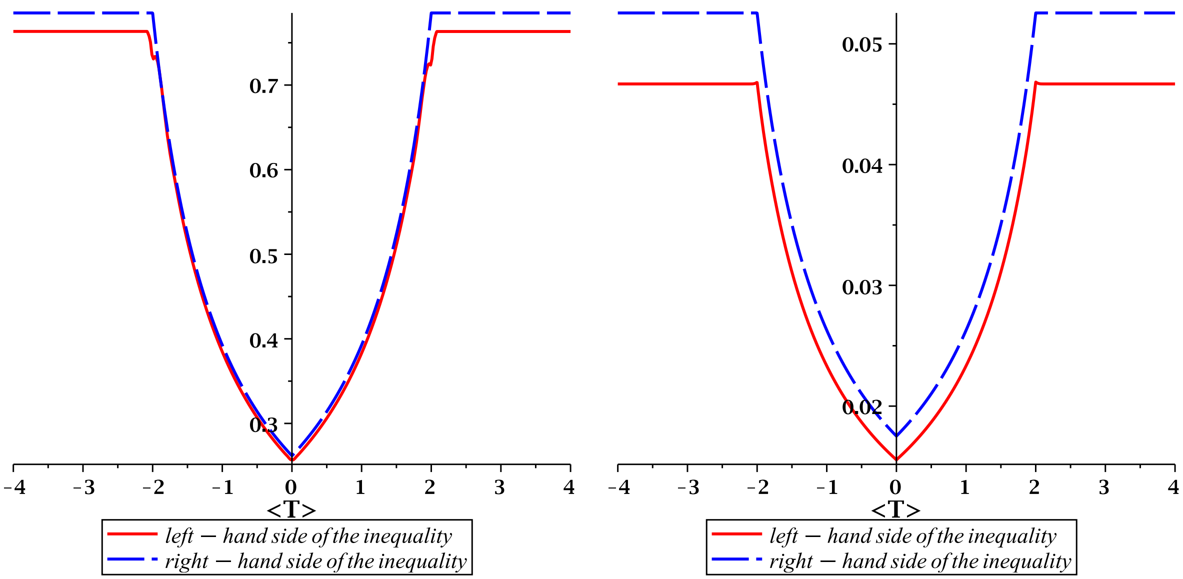
Effective conductivity tensor



Hashin-Shtrikman bounds

$$tr\left[(\Lambda_e(T)-\lambda(T)I)^{-1}\right] \le \frac{1}{\Lambda_2(T)-\lambda(T)} + \frac{1}{\Lambda_1(T)-\lambda(T)},$$

$$tr\left[(\lambda_k(T)I-\Lambda_e(T))^{-1}\right] \le \frac{1}{\lambda_k(T)-\Lambda_2(T)} + \frac{1}{\lambda_k(T)-\Lambda_1(T)},$$



Galka A., Telega, J.J., Tokarzewski, S.: Heat equation with temperature-dependent conductivity coefficients and macroscopic properties of microheterogeneous media, *Mathematical and Computer Modelling*, 33, 927–942, 2001.

Comparison with the results for a random composite

Sevostianov, I., Mishuris, G.: Effective thermal conductivity of a composite with thermo-sensitive constituents and related problems. *International Journal of Engineering Science* 80, 124-135, 2014.

The thermal effective conductivity λ_e of an isotropic composite with temperature dependent and proportional conductivities of the components may be computed by the standard homogenization techniques in the following form:

$$\Lambda_e(T) = \lambda(T) \cdot \Lambda_e, \qquad (13)$$

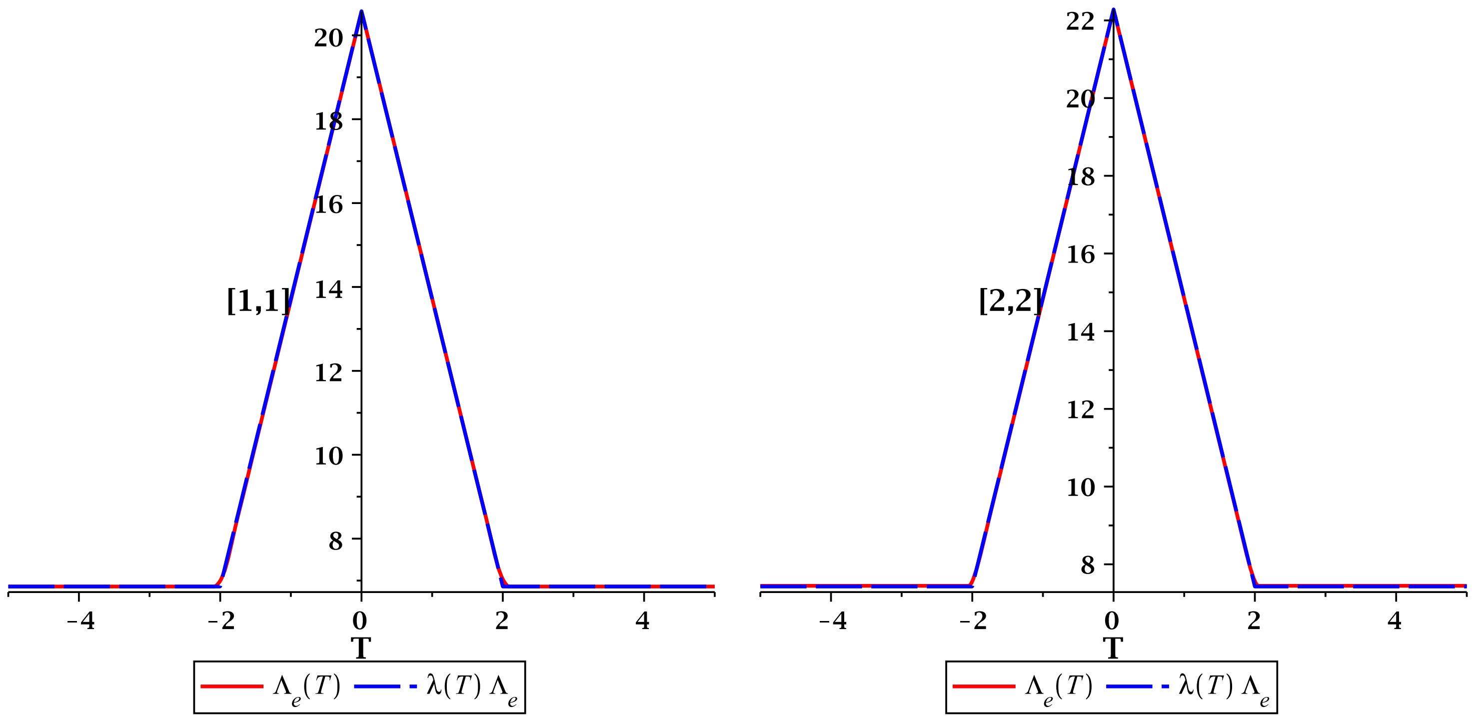
where Λ_e is the effective conductivity tensor of a linear problem with unit conductivity of the matrix and the same ratio C_k between the conductivities of the matrix and inclusions.

Thus, in order to find the effective conductivity for a such kind of composites it is sufficient to find only the effective conductivity of corresponding linear problem. In this particular case we found

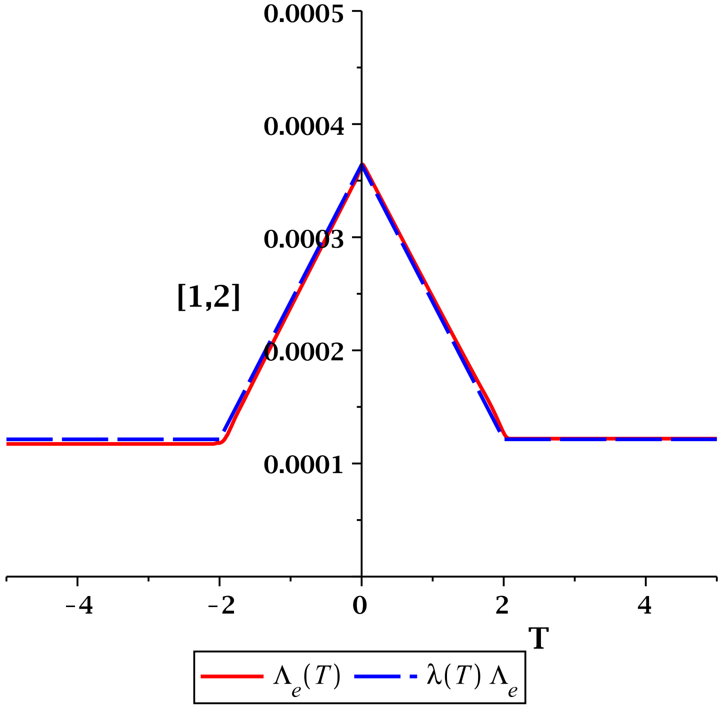
$$\Lambda_e = \begin{pmatrix} 1.524131 & 0.000027 \\ 0.000027 & 1.650632 \end{pmatrix} \qquad (14)$$

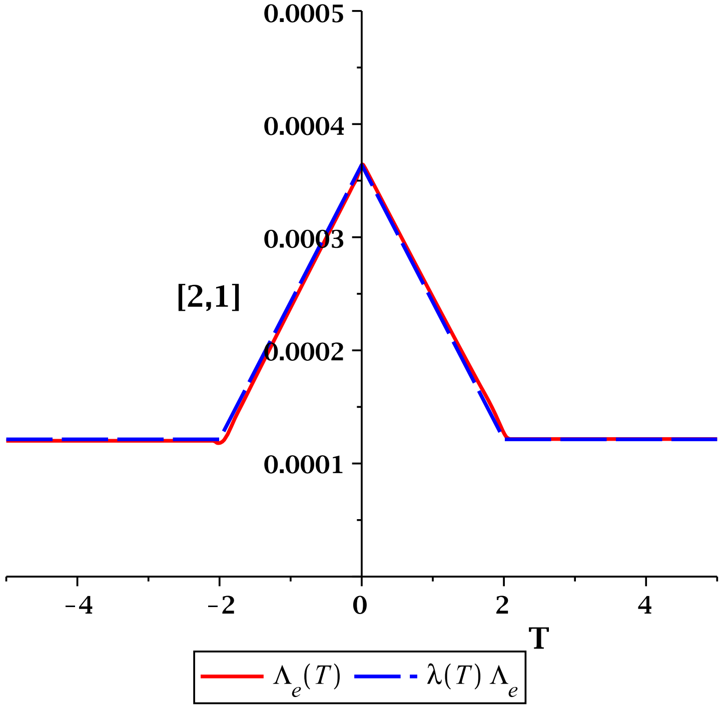
with the same accuracy 10^{-6} as discussed above.

Comparison of the diagonal components of the tensor



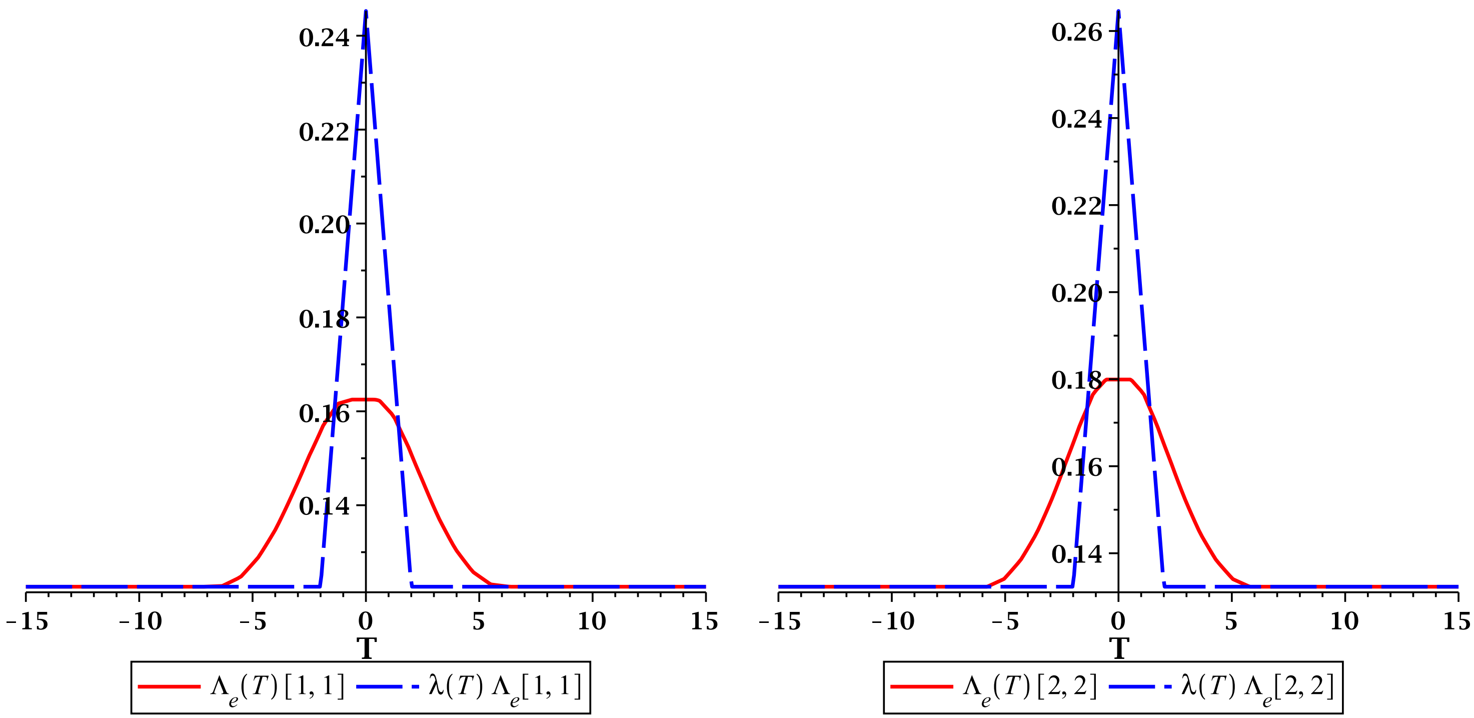
Comparison of the antidiagonal components of the tensor



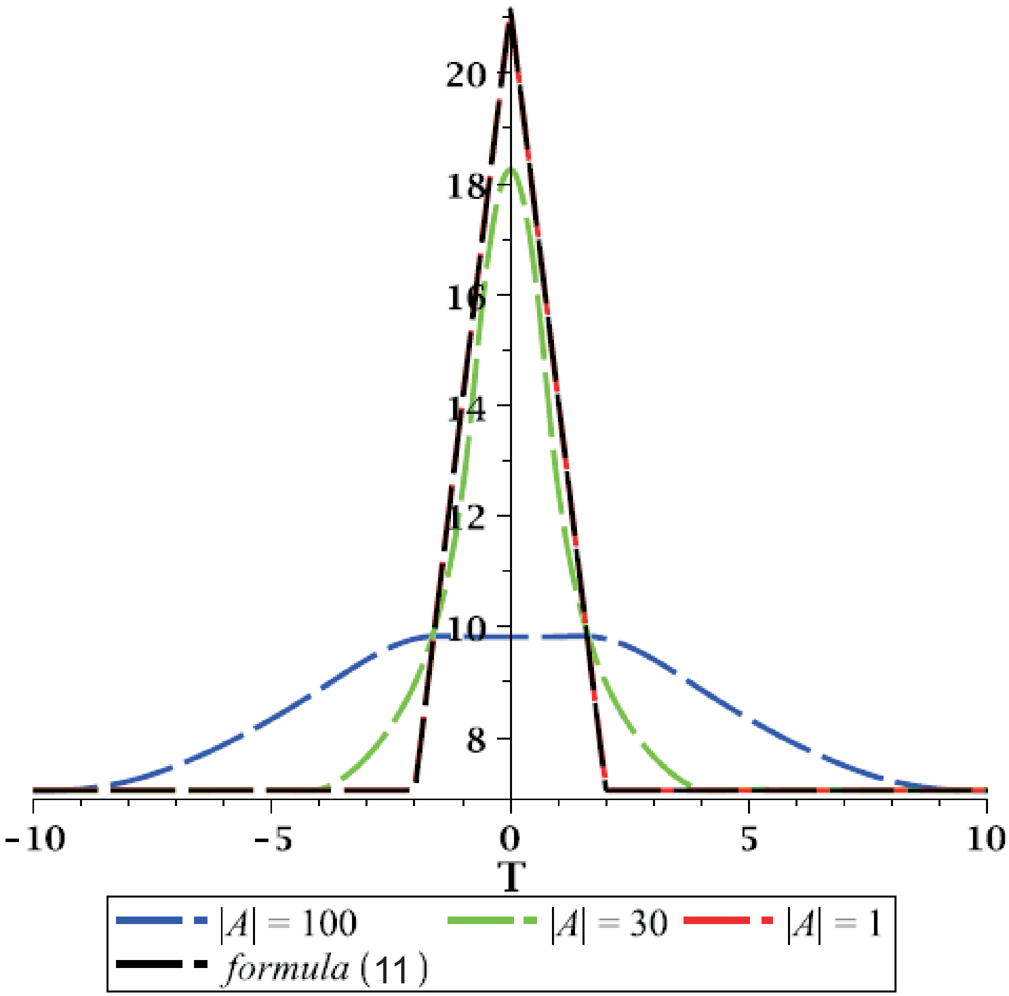


We should make accurate analysis how the flux intensity may impact on the obtained results.

For a high flux intensity, the temperature may change dramatically within one cell, and many unexpected complications appear in non-linear analysis. For instance, comparison of results for periodic and random models shows a large divergence.



The main diagonal components ([1,1] and [2,2]) of the tensors $\Lambda_e(T)$ and $\lambda(T) \cdot \Lambda_e$.



Comparison of the diagonal components of the tensor $\lambda(\langle T_n \rangle) \cdot \Lambda_l(C^{-1}, 1)$ corresponding to the respective linear composite to the diagonal components of the effective conductivity tensors Λ_n of the nonlinear model, for different intensities of the flux.

Conclusions

In the special case of proportional conductivities, we have shown that the effective properties of the representative cell of a nonlinear temperature dependent composite computed by the standard definition (8) may essentially depend on the intensity of the flux penetrating the composite. We have proved that, for a sufficiently small flux intensity, the result for such a nonlinear composite is indistinguishable from another obtained in paper of Sevostianov, I. and Mishuris, G. But this is not the case when the flux intensity becomes sufficiently large. In other words, it is of crucial importance to analyse at the postprocessing stage the level of heat flux locally developed in different parts of the nonlinear composite. In those parts of the composite where the flux is high enough, the average properties cannot be assumed without taking into account a specific structure of the composite and the flux level.

Thank you for attention!