Spatial bounds for resolvent families and applications to PDE'S with critical nonlinearities

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1 Historical Motivation

2 The fractional Cauchy problem with memory effects

3 Uniform Stability

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2 The fractional Cauchy problem with memory effects

3 Uniform Stability

First order case

- We consider

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), \quad t \in (0, \tau] \\ \\ x(0) = x_0 \in D(A), \end{cases} \tag{1}$$

where X is a Banach space, $-A : D(A) \to X$ is a sectorial linear operator of angle $0 \leq \theta < \pi/2$.

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where X is a Banach space, $-A : D(A) \to X$ is a sectorial linear operator of angle $0 \leq \theta < \pi/2$.

- In fact, if f is time independent, it is well known that if $f : X^1 \to X^\alpha$ $(0 < \alpha \leq 1)$ such that

$$\|f(x) - f(y)\|_{X^\alpha} \leq C(R)\|x - y\|_{X^1}, \quad \alpha > 0,\ \|x\|_{X^1}, \|y\|_{X^1} \leq R,$$

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  then (1) is locally well posed.

- $X^\alpha := D((-A)^\alpha)$ and $\|x\|_{X^\alpha} := \|(-A)^\alpha x\|$.

- Let $T : K \to K$ with

$$K(\tau,\mu) = \{x \in C([0,\tau], X^1);\, x(0) = x_0,\ \|x\|_\infty \leq \|x_0\|_{X^1} + \mu)\},$$

where

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$$\|(Tx)(t)\|_{X^1} \leq \|e^{tA}x_0\|_{X^1} + M\int_0^t (t-s)^{\alpha-1}\, ds(\|f(0)\|_{X^\alpha} + C \sup_{0\leq s\leq t}\{\|x(s))\|_{X^1}\}),$$

$$\|(Tx)(t) - (Ty)(t)\|_{X^1} \leq CM\int_0^t (t-s)^{\alpha-1}\, ds \sup_{0\leq s\leq t}\{\|x(s)-y(s)\|_{X^1}\},$$

where it is used that $\|e^{tA}x_0\|_{X^{1-\alpha}} \leq Mt^{\alpha-1}\|x_0\|, \quad t > 0.$

Example

$$\begin{cases} u_t = \Delta u + u|u|^{\rho-1}, & \text{in } \Omega \subset \mathbb{R}^3, \\ u = 0 & \text{in } \partial\Omega, \\ u(0) = u_0. \end{cases}$$

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Δ is an unbounded operator on $X = H^{-1}(\Omega) := (E^{1/2})'$, where $E^{1/2}$ is the fractional space associated to Δ in $L^2(\Omega)$ with Dirichlet boundary conditions, with domain $X^1 := H_0^1(\Omega)$, and

$$X^\alpha \hookrightarrow H^{2\alpha-1}, \quad \alpha > 1/2,$$

$$X^{1/2} = L^2(\Omega),$$

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For $1 < \rho < 5$, $f : X^1 \to X^\alpha$ for some $0 < \alpha < 1$. For $\rho = 5$, $f : X^1 \to X$, and we are in the critical case.

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For $1 < \rho < 5$, $f : X^1 \to X^\alpha$ for some $0 < \alpha < 1$. For $\rho = 5$, $f : X^1 \to X$, and we are in the critical case.

But for $\rho = 5$, by using the Sobolev embeddings, if ϵ is small then $f : X^{1+\epsilon} \to X^{5\epsilon}$, while $A : X^{1+\epsilon} \to X^\epsilon$.

ε-regular map For $\varepsilon > 0$ we say that a map g is ε-regular relative to (X^1, X) if there exist $\rho > 1, \gamma(\varepsilon)$ with $\rho\varepsilon \le \gamma(\varepsilon) < 1$, and $c > 0$ such that $g : X^{1+\varepsilon} \to X^{\gamma(\varepsilon)}$ satisfying

$$\|g(x)-g(y)\|_{X^{\gamma(\varepsilon)}} \le c(1+\|x\|^{\rho-1}_{X^{1+\varepsilon}}+\|y\|^{\rho-1}_{X^{1+\varepsilon}})\|x-y\|_{X^{1+\varepsilon}}, \quad x, y \in X^{1+\varepsilon}.$$

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The class $\mathcal{F}(\nu)$ Let $\varepsilon, \gamma(\varepsilon), \xi, \zeta, c, \delta' > 0$, and a real function ν such that $0 \leq \nu(t) < \delta'$ and $\lim_{t\to 0^+} \nu(t) = 0$. The class $\mathcal{F}(\varepsilon, \gamma(\varepsilon), c, \nu, \xi, \zeta)$ denotes the family of functions f such that, for $t \geq 0$ $f(t, \cdot)$ is an ε-regular map relative to (X^1, X), satisfying for all $x, y \in X^{1+\varepsilon}$

$$\|f(t,x) - f(t,y)\|_{X^{\gamma(\varepsilon)}} \leq c(\|x\|^{\rho-1}_{X^{1+\varepsilon}} + \|y\|^{\rho-1}_{X^{1+\varepsilon}} + \nu(t)t^{-\zeta})\|x-y\|_{X^{1+\varepsilon}},$$

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$$\begin{aligned} \|f(t,x) - f(t,y)\|_{X^{\gamma(\varepsilon)}} &\leq c(\|x\|^{\rho-1}_{X^{1+\varepsilon}} + \|y\|^{\rho-1}_{X^{1+\varepsilon}} + \nu(t)t^{-\zeta})\|x - y\|_{X^{1+\varepsilon}}, \\ \|f(t,x)\|_{X^{\gamma(\varepsilon)}} &\leq c(\|x\|^{\rho}_{X^{1+\varepsilon}} + \nu(t)t^{-\xi}). \end{aligned}$$

ε-regular mild solution We say that $x : [0, \tau] \to X^1$ is an ε-regular mild solution to (1) if $x \in C([0, \tau], X^1) \cap C((0, \tau], X^{1+\varepsilon})$ and

$$x(t) = e^{tA}x_0 + \int_0^t e^{A(t-s)} f(s, x(s))ds.$$

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Specifically, fractional models allow to describe phenomena on viscous fluids or in special types of porous medium.

- Let

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t)), \quad t \in (0, \tau], \\ \\ x(0) = x_0 \end{cases} \tag{2}$$

where $0 < \alpha \leq 1$, D_t^α is the Caputo fractional derivative, $-A : D(A) \to X$ is a sectorial operator and f belongs to the class $\mathcal{F}(\nu)$.

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- Let $(R_\alpha(t))_{t>0}$ and $(S_\alpha(t))_{t\geq 0}$ defined by
$$R_\alpha(t) := \frac{1}{2\pi i} \int_\gamma e^{\lambda t} (\lambda^\alpha - A)^{-1} d\lambda, \quad t > 0,$$
and
$$S_\alpha(t) := \frac{1}{2\pi i} \int_\gamma e^{\lambda t} \lambda^{\alpha - 1} (\lambda^\alpha - A)^{-1} d\lambda, \quad t > 0,$$
where $\gamma \subset \rho(A)$ is a suitable Hankel's path.

ε-regular mild solution We say that $x : [0, \tau] \to X^1$ is an ε-regular mild solution to (2) if $x \in C([0, \tau], X^1) \cap C((0, \tau], X^{1+\varepsilon})$ and

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The resolvent and integral resolvent for (2) generated by A satisfy

$$\|S_\alpha(t)x\|_{X^{1+\theta}} \leq Mt^{-\alpha(1+\theta-\beta)}\|x\|_{X^\beta}, \quad x \in X^\beta,$$

$$\|R_\alpha(t)x\|_{X^{1+\theta}} \leq Mt^{-\alpha(\theta-\beta)-1}\|x\|_{X^\beta}, \quad x \in X^\beta,$$

for all $0 \leq \theta, \beta \leq 1$.

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- However, in some cases, the memory of the model depends on the operator that governs the problem, mainly on viscous fluids or in the theory of heat conduction when inner heat sources are of special types.
- First and second order abstract problems with memory terms.
- Moore-Gibson-Thompson with memory.

Let $0 < \alpha \leq 1$ and

$$\begin{cases} {}_C D_t^{\alpha} x(t) - Ax(t) + \int_0^t \beta(t-s) Ax(s)\, ds = f(t, x(t)), \quad t \in (0, \tau], \\ x(0) = x_0 \in X, \end{cases} \tag{3}$$

where $-A$ is a sectorial linear operator of angle $0 \leq \theta < \pi/2$ on X, and the memory kernel β is given by

$$\beta(t) := e^{-\delta t} g_{\nu}(t) = e^{-\delta t} \frac{t^{\nu-1}}{\Gamma(\nu)}, \quad t > 0,\ 0 < \nu \leq 1,\ \delta \geq 0.$$

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- The convolution term $\int_0^t \beta(t-s) Ax(s)\, ds$ reflects the memory effect of viscoelastic materials.
- In the memory term $\int_0^t \beta(t-s) Ax(s)\, ds$, Ax represents the background of deformations, β is called the relaxation function and $\int_0^t \beta(s)\, ds$ is the intensity of the memory.

Supposing that $x : [0, \infty) \to X$ satisfies (3) and it is of subexponential growth,

$$\lambda^{\alpha}\hat{x}(\lambda) - \lambda^{\alpha-1}x_0 - A\hat{x}(\lambda) + A\hat{\beta}(\lambda)\hat{x}(\lambda) = \widehat{f(\cdot, x(\cdot))}(\lambda),$$

with $\hat{\beta}(\lambda) = \frac{1}{(\lambda+\delta)^{\nu}}$.

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If $\frac{\lambda^{\alpha}(\lambda+\delta)^{\nu}}{(\lambda+\delta)^{\nu}-1} \in \rho(A)$, then

$$\begin{aligned} \hat{x}(\lambda) &= \frac{\lambda^{\alpha-1}(\lambda+\delta)^{\nu}}{(\lambda+\delta)^{\nu}-1}\left(\frac{\lambda^{\alpha}(\lambda+\delta)^{\nu}}{(\lambda+\delta)^{\nu}-1} - A\right)^{-1} x_0 \\ &\quad + \frac{(\lambda+\delta)^{\nu}}{(\lambda+\delta)^{\nu}-1}\left(\frac{\lambda^{\alpha}(\lambda+\delta)^{\nu}}{(\lambda+\delta)^{\nu}-1} - A\right)^{-1} \widehat{f(\cdot, x(\cdot))}(\lambda). \end{aligned}$$

Theorem

(i) *If* $\delta \geq 1$ *and* $t > 0$,

$$S(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\frac{\lambda^{\alpha-1}(\lambda+\delta)^\nu}{(\lambda+\delta)^\nu - 1}\left(\frac{\lambda^\alpha(\lambda+\delta)^\nu}{(\lambda+\delta)^\nu - 1} - A\right)^{-1} d\lambda,$$

$$R(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\frac{(\lambda+\delta)^\nu}{(\lambda+\delta)^\nu - 1}\left(\frac{\lambda^\alpha(\lambda+\delta)^\nu}{(\lambda+\delta)^\nu - 1} - A\right)^{-1} d\lambda,$$

where $\gamma \subset \rho(A)$. *Furthermore* $\|S(t)\| \leq M$ *for* $t \geq 0$ *and* $\|R(t)\| \leq M t^{\alpha-1}$ *for* $t > 0$.

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where $\gamma \subset \rho(A)$. *Furthermore* $\|S(t)\| \leq M$ *for* $t \geq 0$ *and* $\|R(t)\| \leq Mt^{\alpha-1}$ *for* $t > 0$.

(ii) *If* $0 \leq \delta < 1$ *and* $t > 0$,

$$S(t) = \frac{1}{2\pi i}\int_{1-\delta+\gamma} e^{\lambda t}\frac{\lambda^{\alpha-1}(\lambda+\delta)^{\nu}}{(\lambda+\delta)^{\nu}-1}\left(\frac{\lambda^{\alpha}(\lambda+\delta)^{\nu}}{(\lambda+\delta)^{\nu}-1} - A\right)^{-1} x\, d\lambda,$$

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where $\gamma \subset \rho(A)$. *Furthermore* $\|S(t)\| \leq Me^{(1-\delta)t}$ *for* $t \geq 0$ *and* $\|R(t)\| \leq Mt^{\alpha-1}e^{(1-\delta)t}$ *for* $t > 0$.

Proof

Let $\delta \geq 1$.

- For $t > 0$ we take $r = 1/t$, $\omega \in (\pi/2, \pi - \theta)$, and $\gamma_{r,\omega} = \{\lambda e^{i\omega} : \lambda \geq r\} \cup \{re^{i\varphi} : \varphi \in (-\omega, \omega)\} \cup \{\lambda e^{-i\omega} : \lambda \geq r\} := \gamma_1 \cup \gamma_2 \cup \gamma_3$, oriented counterclockwise.

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- If $\lambda \in \gamma_{r,\omega}$,

$$-\alpha\omega \leq arg\left(\frac{\lambda^\alpha(\lambda+\delta)^\nu}{(\lambda+\delta)^\nu - 1}\right) \leq \omega, \qquad arg(\lambda) \geq 0,$$

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  so $\frac{\lambda^\alpha(\lambda+\delta)^\nu}{(\lambda+\delta)^\nu-1} \in \Sigma_\omega \subset \rho(A)$.

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  so $\frac{\lambda^\alpha(\lambda+\delta)^\nu}{(\lambda+\delta)^\nu-1} \in \Sigma_\omega \subset \rho(A)$.

- We get $\|S(t)\| \leq M$ for $t \geq 0$ and $\|R(t)\| \leq Mt^{\alpha-1}$ for $t > 0$, working separately in γ_1, γ_2 and γ_3.

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- Let $x \in D(A)$,

$$
\begin{aligned}
&\|S(t)x - x\| \\
&= \|\tfrac{1}{2\pi i}\int_{\gamma_{r,\omega}} e^{\lambda t}\left(\tfrac{\lambda^{\alpha-1}(\lambda+\delta)^{\nu}}{(\lambda+\delta)^{\nu}-1}\left(\tfrac{\lambda^{\alpha}(\lambda+\delta)^{\nu}}{(\lambda+\delta)^{\nu}-1} - A\right)^{-1} - \lambda^{-1}\right)x\,d\lambda\| \\
&= \|\frac{1}{2\pi i}\int_{\gamma_{r,\omega}} e^{\lambda t}\left(\frac{\lambda^{\alpha}(\lambda+\delta)^{\nu}}{(\lambda+\delta)^{\nu}-1} - A\right)^{-1}\lambda^{-1}Ax\,d\lambda\| \\
&\leq \frac{M_{\omega}\|Ax\|}{2\pi}\int_{\gamma_{r,\omega}}\left|e^{\lambda t}\lambda^{-1-\alpha}\left(\frac{(\lambda+\delta)^{\nu}-1}{(\lambda+\delta)^{\nu}}\right)d\lambda\right| \leq M\|Ax\|t^{\alpha} \to 0,
\end{aligned}
$$

as $t \to 0^+$.

The operator families satisfy the Volterra integral equations

$$S(t)x = x + \int_0^t a(t-s)AS(s)x\,ds \quad x \in D(A),\, t \geq 0,$$

$$R(t)x = g_\alpha(t)x + \int_0^t a(t-s)AR(s)xds, \quad x \in D(A),\, t > 0,$$

where $a(t) := g_\alpha(t) - (g_\alpha * \beta)(t)$.

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It is an easy computation that

$$S(t)x := (g_{1-\alpha} * R)(t)x = \int_0^t g_{1-\alpha}(t-s)R(s)x\,ds, \quad x \in X, \quad t > 0.$$

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(i) *If* $\delta \geq 1$,

$$\|S(t)x\|_{X^\beta} \leq Mt^{-\alpha\beta}\|x\|, \quad \|R(t)x\|_{X^\beta} \leq Mt^{\alpha(1-\beta)-1}\|x\|,$$

for $t > 0$ *and* $x \in X$.

Theorem

Let $0 \leq \beta \leq 1$.

(i) *If* $\delta \geq 1$,

$$\|S(t)x\|_{X^\beta} \leq Mt^{-\alpha\beta}\|x\|, \quad \|R(t)x\|_{X^\beta} \leq Mt^{\alpha(1-\beta)-1}\|x\|,$$

for $t > 0$ *and* $x \in X$.

(ii) *If* $0 \leq \delta < 1$,

$$\|S(t)x\|_{X^\beta} \leq Me^{(1-\delta)t}t^{-\alpha\beta}\|x\|, \quad \|R(t)x\|_{X^\beta} \leq Me^{(1-\delta)t}t^{\alpha(1-\beta)-1}\|x\|,$$

for $t > 0$ *and* $x \in X$.

Linear and non-linear case

Mild solution We say that $x \in C([0,\tau]; X)$ is a mild solution of (3) if

$$x(t) = S(t)x_0 + \int_0^t R(t-s) f(s, x(s)) ds.$$

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Strong solution Let $f : \mathbb{R}_+ \to X$ be continuous such that $f \in W^{1,1}_{loc}(\mathbb{R}_+; X)$, and $x_0 \in D(A)$. Then, the problem (3) has a unique global strong solution, that is, such solution $x \in C([0,\infty); D(A)) \cap C^1([0,\infty); X)$ and satisfies (3).

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Blow up Theorem Let $f : [0,\infty) \times X \to X$ under suitable locally Lipschitz conditions. Then either (3) has a global mild solution or there exists $\omega > 0$ such that $x : [0,\omega) \to X$ is a maximal local mild solution with $\lim_{t\to\omega^-} \|x(t)\| = \infty$.

Theorem

Let $f \in \mathcal{F}(\nu)$ *and* $y_0 \in X^1$. *There exist* $r, \tau > 0$ *such that for any* $x_0 \in B_{X^1}(y_0, r)$ *there is* $x(\cdot, x_0) \in C([0,\tau]; X^1)$ *with* $x(0, x_0) = x_0$ *which is an* ε*-regular mild solution to* (3)*. This solution satisfies*

$$x(\cdot, x_0) \in C((0,\tau], X^{1+\theta}), \quad 0 \leq \theta < \gamma(\varepsilon),$$

and for $0 < \theta < \gamma(\varepsilon)$,

$$\lim_{t \to 0^+} t^{\alpha\theta} \|x(t, x_0)\|_{X^{1+\theta}} = 0, \quad \delta \geq 1,$$

$$\lim_{t \to 0^+} t^{\alpha\theta} e^{(\delta-1)t} \|x(t, x_0)\|_{X^{1+\theta}} = 0, \quad 0 \leq \delta < 1.$$

Moreover, for each $\theta_0 < \gamma(\varepsilon) + \varepsilon - \rho\varepsilon$ *there exists* $C > 0$ *such that if* $x_0, z_0 \in B_{X^1}(y_0, r)$*, then*

$$t^{\alpha\theta} \|x(t, x_0) - x(t, z_0)\|_{X^{1+\theta}} \leq C \|x_0 - z_0\|_{X^1}, \quad \delta \geq 1,$$

$$t^{\alpha\theta} e^{(\delta-1)t} \|x(t, x_0) - x(t, z_0)\|_{X^{1+\theta}} \leq C \|x_0 - z_0\|_{X^1}, \quad 0 \leq \delta < 1,$$

for $t \in [0, \tau]$*, and* $0 \leq \theta \leq \theta_0$.

1 Historical Motivation

2 The fractional Cauchy problem with memory effects

3 Uniform Stability

Theorem

Let $-A$ *be a* $-a$*-sectorial of angle* $\vartheta \in [0, \pi/2)$ *with* $a > 0$, *and* $0 \leq \beta < 1$. *For* $x \in X$ *it follows*

(i) $\|S_\alpha(t)x\|_{X^\beta} \leq M e^{1-\beta}_{\alpha, 1-\alpha\beta}(t, a)\|x\|, \quad t > 0.$

(ii) $\|R_\alpha(t)x\|_{X^\beta} \leq M e^{1-\beta}_{\alpha, \alpha(1-\beta)}(t, a)\|x\|, \quad t > 0.$

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Thank you for your attention

Questions?