Semi-Fredholm theory for singular integral operators with shifts and slowly oscillating data

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IWOTA, Chemnitz, August 14-18, 2017

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Left and right Fredholm operators

Notation:

- X is a Banach space
- $\mathcal{B}(X)$ is the Banach algebra of all bounded linear operators on the space X
- $\mathcal{K}(X)$ is the closed two-sided ideal of all compact operators on the space X
- $\mathcal{B}^\pi(X) := \mathcal{B}(X)/\mathcal{K}(X)$ is the Calkin algebra of the cosets

$$A^\pi := A + \mathcal{K}(X) \quad \text{where} \quad A \in \mathcal{B}(X).$$

An operator $A \in \mathcal{B}(X)$ is said to be

left Fredholm / right Fredholm

if the coset A^π is

left invertible / right invertible

in the Calkin algebra $\mathcal{B}^\pi(X)$.

n-normal and d-normal operators

An operator $A \in \mathcal{B}(X)$ is said to be n-normal / d-normal on X if its image $\operatorname{Im} A$ is closed and

$$n(A) := \dim \operatorname{Ker} A < \infty \quad / \quad d(A) := \dim(X / \operatorname{Im} A) < \infty.$$

Theorem

If X is a Banach space, then

$$\begin{aligned} A \text{ is left Fredholm} &\Rightarrow A \text{ is } n\text{-normal} \\ A \text{ is right Fredholm} &\Rightarrow A \text{ is } d\text{-normal} \end{aligned}$$

If X is a Hilbert space, then

$$\begin{aligned} A \text{ is left Fredholm} &\Leftrightarrow A \text{ is } n\text{-normal} \\ A \text{ is right Fredholm} &\Leftrightarrow A \text{ is } d\text{-normal} \end{aligned}$$

Fredholm and semi-Fredholm operators

An operator A is said to be Fredholm if it is

- left and right Fredholm,
- equivalently, n-normal and d-normal

The index of a Fredholm operator A is defined by

$$\operatorname{Ind} A = n(A) - d(A).$$

An operator A is said to be semi-Fredholm if it is n-normal or d-normal.

The weighted Cauchy singular integral operator

Theorem (Boris Khvedelidze, 1956)

Let $1 < p < \infty$ and $\gamma \in \mathbb{C}$ be such that

$$0 < 1/p + \Re\gamma < 1.$$

Then the weighted Cauchy singular integral operator S_γ given by

$$(S_\gamma f)(t) := \frac{1}{\pi i} p.v. \int_{\mathbb{R}_+} \left(\frac{t}{\tau}\right)^\gamma \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \mathbb{R}_+,$$

is bounded on the Lebesgue space $L^p(\mathbb{R}_+)$.

Notation:

$$P_\gamma^\pm = (I \pm S_\gamma)/2.$$

Warning:

$$(P_\gamma^\pm)^2 \neq P_\gamma^\pm.$$

Aim of the work

Find criteria for n-normality / d-normality on $L^p(\mathbb{R}_+)$ of the paired operator of the form

$$N = A_+ P_\gamma^+ + A_- P_\gamma^-,$$

where $A_\pm$ are functional operators with shifts and slowly oscillating data.

Slowly oscillating functions (Sarason, 1977)

A bounded continuous function f on $\mathbb{R}_+ = (0, \infty)$ is called *slowly oscillating* (at 0 and ∞) if for each (equivalently, for some) $\lambda \in (0, 1)$,

$$\lim_{r \to s} \underbrace{\left(\sup_{t,\tau \in [\lambda r, r]} |f(t) - f(\tau)| \right)}_{\text{oscillation}} = 0 \quad \text{for} \quad s \in \{0, \infty\}.$$

The set $SO(\mathbb{R}_+)$ of all slowly oscillating functions forms a C^*-algebra and

$$C(\overline{\mathbb{R}}_+) \subset SO(\mathbb{R}_+), \quad C(\overline{\mathbb{R}}_+) \neq SO(\mathbb{R}_+),$$

where $C(\overline{\mathbb{R}}_+)$ is the set of all continuous functions on

$$\overline{\mathbb{R}}_+ = [0, +\infty].$$

Slowly oscillating shifts

Suppose α is an orientation-preserving homeomorphism of $[0, \infty]$ itself, which has only two fixed points 0 and ∞ and suppose that its restriction to $\mathbb{R}_+$ is a diffeomorphism.

We say that α is a *slowly oscillating shift* if

- $\log \alpha'$ is bounded,
- $\alpha' \in SO(\mathbb{R}_+)$.

The set of all slowly oscillating shifts is denoted by $SOS(\mathbb{R}_+)$.

Trivial example:
Let $c \in \mathbb{R}_+ \setminus \{1\}$ and $\alpha(t) = ct$. Then $\alpha \in SOS(\mathbb{R}_+)$.

Non-trivial examples of slowly oscillating shifts can be constructed with the aid of the following lemma.

Exponent function of a slowly oscillating shift

Lemma (KKL, 2011)

Suppose α is an orientation-preserving homeomorphism of $[0, \infty]$ itself, which has only two fixed points 0 and ∞ and suppose that its restriction to $\mathbb{R}_+$ is a diffeomorphism. Then $\alpha \in SOS(\mathbb{R}_+)$ if and only if

$$\alpha(t) = te^{\omega(t)}, \quad t \in \mathbb{R}_+,$$

for some real-valued function $\omega \in SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ such that the function $t \mapsto t\omega'(t)$ also belongs to $SO(\mathbb{R}_+)$ and

$$\inf_{t \in \mathbb{R}_+} \big(1 + t\omega'(t)\big) > 0.$$

The real-valued slowly oscillating function

$$\omega(t) = \log[\alpha(t)/t]$$

is called the exponent function of $\alpha \in SOS(\mathbb{R}_+)$.

Shift operator

We suppose that $1 < p < \infty$ and consider the shift operator U_α defined by

$$U_\alpha f = (\alpha')^{1/p} f \circ \alpha.$$

It is easy to see that $U_\alpha \in \mathcal{B}(L^p(\mathbb{R}_+))$ and U_α is an isometry whenever $\alpha \in SOS(\mathbb{R}_+)$.

Wiener algebra of functional operators

Let $\alpha \in SOS(\mathbb{R}_+)$. For $k \in \mathbb{N}$, put

$$U_\alpha^{-k} := (U_\alpha^{-1})^k.$$

Denote by $W_{\alpha,p}^{SO}$ the collection of all operators of the form

$$A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k$$

where $a_k \in SO(\mathbb{R}_+)$ for all $k \in \mathbb{Z}$ and

$$\|A\|_W := \sum_{k\in\mathbb{Z}} \|a_k\|_{C_b(\mathbb{R}_+)} < +\infty. \tag{1}$$

The set $W_{\alpha,p}^{SO}$ is a Banach algebra with respect to the usual operations and the norm (1).

By analogy with the Wiener algebra of absolutely convergent Fourier series, we will call $W_{\alpha,p}^{SO}$ the Wiener algebra.

Brief history of the study of $A_+P_\gamma^+ + A_-P_\gamma^-$:

1. no shifts, continuous data, Fredholm and semi-Fredholm theory
   Israel Gohberg, Naum Krupnik, 1970's
2. continuous data, Fredholm theory
   Yuri Karlovich, Viktor Kravchenko, 1981
3. continuous data, semi-Fredholm theory
   Yuri Karlovich, Rasul Mardiev, 1985
4. no shifts, slowly oscillating data, Fredholm theory
   Albrecht Böttcher, Yuri Karlovich, Vladimir Rabinovich, 1990–2000
5. binomial functional operators A_+ and A_- with shifts and slowly oscillating data, Fredholm theory
   KKL, Fredholm criteria – 2011, an index formula – 2017
6. functional operators A_+ and A_- of Wiener type with shifts and slowly oscillating data, Fredholm criteria
   Gustavo Fernandéz-Torres and Yuri Karlovich, 2016

Theorem (Main result: incomplete form, 2017)

Let $1 < p < \infty$ *and let* $\gamma \in \mathbb{C}$ *satisfy* $0 < 1/p + \Re\gamma < 1$. *Suppose*

$$a_k, b_k \in SO(\mathbb{R}_+) \text{ for all } k \in \mathbb{Z}, \quad \alpha, \beta \in SOS(\mathbb{R}_+),$$

$$A_+ = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in W_{\alpha,p}^{SO}, \quad A_- = \sum_{k\in\mathbb{Z}} b_k U_\beta^k \in W_{\beta,p}^{SO}.$$

For the operator

$$N = A_+ P_\gamma^+ + A_- P_\gamma^-,$$

the following assertions are equivalent:

(a) *the operator* N *is n-normal / d-normal on the space* $L^p(\mathbb{R}_+)$,

(b) *the operator* N *is left Fredholm / right Fredholm on* $L^p(\mathbb{R}_+)$,

(c) *the following two conditions are fulfilled:*

- (c-i) *the operators* A_+ *and* A_- *are left invertible / right invertible on the space* $L^p(\mathbb{R}_+)$;
- (c-ii) *the function* n *(will be defined later) associated to the operator* N *does not vanish in a certain sense.*

Corollary (Fredholm criterion, 2016)

Let $1 < p < \infty$ *and let* $\gamma \in \mathbb{C}$ *satisfy* $0 < 1/p + \Re\gamma < 1$*. Suppose*

$$a_k, b_k \in SO(\mathbb{R}_+) \text{ for all } k \in \mathbb{Z}, \quad \alpha, \beta \in SOS(\mathbb{R}_+),$$

$$A_+ = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in W_{\alpha,p}^{SO}, \quad A_- = \sum_{k\in\mathbb{Z}} b_k U_\beta^k \in W_{\beta,p}^{SO}.$$

For the operator

$$N = A_+ P_\gamma^+ + A_- P_\gamma^-,$$

the following assertions are equivalent:

(a) (=(b)) *the operator* N *is Fredholm on the space* $L^p(\mathbb{R}_+)$*,*

(c) *the following two conditions are fulfilled:*

- (c-i) *the operators* A_+ *and* A_- *are invertible on the space* $L^p(\mathbb{R}_+)$*;*
- (c-ii) *the same as in the main theorem.*

An index formula is available for the case

$$A_+ = a_0 I + a_1 U_\alpha, \quad A_- = b_0 I + b_1 U_\beta.$$

Invertibility of binomial functional operators

Let $a, b \in SO(\mathbb{R}_+)$. We say that a dominates b and write $a \gg b$ if

$$\inf_{t\in\mathbb{R}_+} |a(t)| > 0, \quad \liminf_{t\to s}(|a(t)| - |b(t)|) > 0, \quad s \in \{0, \infty\}.$$

Theorem (KKL, 2011, 2016
for continuous data - Viktor Kravchenko, 1974)

Suppose $a, b \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$. The binomial functional operator $aI - bU_\alpha$ is invertible on the Lebesgue space $L^p(\mathbb{R}_+)$ if and only if either $a \gg b$ or $b \gg a$.

(a) *If $a \gg b$, then* $$(aI - bU_\alpha)^{-1} = \sum_{n=0}^{\infty}(a^{-1}bU_\alpha)^n a^{-1}I.$$

(b) *If $b \gg a$, then* $$(aI - bU_\alpha)^{-1} = -U_\alpha^{-1}\sum_{n=0}^{\infty}(b^{-1}aU_\alpha^{-1})^n b^{-1}I.$$

Attracting and repelling points of the shift

Suppose

$$\alpha_0(t) := t, \quad \alpha_n(t) := \alpha[\alpha_{n-1}(t)] \quad \text{for} \quad n \in \mathbb{Z} \quad \text{and} \quad t \in \mathbb{R}_+.$$

Fix a point $\tau \in \mathbb{R}_+$ and put

$$\tau_- := \lim_{n\to-\infty} \alpha_n(\tau), \quad \tau_+ := \lim_{n\to+\infty} \alpha_n(\tau).$$

Then

- either $\tau_- = 0$ and $\tau_+ = \infty$,
- or $\tau_- = \infty$ and $\tau_+ = 0$.

The points τ_+ and τ_- are called attracting and repelling points of the shift α, respectively.

Strict one-sided invertibility of binomial FOs

Theorem (KKL, 2016,
for continuous data - Yuri Karlovich, Mardiev, 1985)

Suppose $a, b \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$. The binomial functional operator $A = aI - bU_\alpha$ is strictly left/right invertible on the space $L^p(\mathbb{R}_+)$ if and only if

$$\limsup_{t\to\tau_-}(|a(t)| - |b(t)|) < 0 < \liminf_{t\to\tau_+}(|a(t)| - |b(t)|)$$

$$\limsup_{t\to\tau_+}(|a(t)| - |b(t)|) < 0 < \liminf_{t\to\tau_-}(|a(t)| - |b(t)|)$$

and for every $t \in \mathbb{R}_+$ there is an integer k_t such that

$$b[\alpha_k(t)] \neq 0 \text{ for } k < k_t \text{ and } a[\alpha_k(t)] \neq 0 \text{ for } k > k_t.$$

$$b[\alpha_k(t)] \neq 0 \text{ for } k \geq k_t \text{ and } a[\alpha_k(t)] \neq 0 \text{ for } k < k_t.$$

If the operator A is strictly left/right invertible, then at least one of its left/right inverses belongs to the Banach algebra $\mathcal{FO}_\alpha^W$.

Mellin convolution operators

Let $d\mu(t) = dt/t$ be the (normalized) invariant measure on $\mathbb{R}_+$ and $\mathcal{M} : L^2(\mathbb{R}_+, d\mu) \to L^2(\mathbb{R})$ be the Mellin transform.

A function $a \in L^\infty(\mathbb{R})$ is called a Mellin multiplier on $L^p(\mathbb{R}_+, d\mu)$ if the mapping

$$f \mapsto \mathcal{M}^{-1} a \mathcal{M} f$$

maps $L^2(\mathbb{R}_+, d\mu) \cap L^p(\mathbb{R}_+, d\mu)$ into itself and extends to a bounded operator $\mathrm{Co}(a)$ on $L^p(\mathbb{R}_+, d\mu)$. The set of all Mellin multipliers is denoted by $\mathcal{M}_p(\mathbb{R})$.

Singular integral operators as Mellin convolution operators

Consider the isometric isomorphism

$$\Phi : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+, d\mu), \quad (\Phi f)(t) := t^{1/p} f(t), \quad t \in \mathbb{R}_+.$$

Lemma (see, e.g., Roch-Santos-Silbermann's book 2011)

Let $1 < p < \infty$ and $\gamma \in \mathbb{C}$ be such that $0 < 1/p + \Re\gamma < 1$. Then the functions

$$p_\gamma^\pm(x) := \frac{1}{2}(1 \pm \coth[\pi(x + i/p + i\gamma)]), \quad x \in \mathbb{R},$$

belong to $\mathcal{M}_p(\mathbb{R})$ and

$$P_\gamma^\pm = \Phi^{-1}\,\mathrm{Co}(p_\gamma^\pm)\Phi.$$

Baby shift operators as Mellin convolution operators

For $\omega, \eta \in \mathbb{R} \setminus \{0\}$, consider the baby slowly oscillating shifts

$$\alpha(t) = te^{\omega}, \quad \beta(t) = te^{\eta}, \quad t \in \mathbb{R}_+,$$

and also recall that an adult slowly oscillating shift is of the form

$$\gamma(t) = te^{\psi(t)} \quad \text{with} \quad \psi \in SO(\mathbb{R}_+).$$

Then the functions

$$e_\omega(x) = e^{i\omega x}, \quad e_\eta(x) = e^{i\eta x}, \quad x \in \mathbb{R},$$

belong to $\mathcal{M}_p(\mathbb{R})$ and

$$U_\alpha = \Phi^{-1}\operatorname{Co}(e_\omega)\Phi, \quad U_\beta = \Phi^{-1}\operatorname{Co}(e_\eta)\Phi.$$

More generally, for all $k \in \mathbb{Z}$,

$$U_\alpha^k = \Phi^{-1}\operatorname{Co}(e_{k\omega})\Phi, \quad U_\beta^k = \Phi^{-1}\operatorname{Co}(e_{k\eta})\Phi.$$

Operator N_{baby} and function n_{baby}

Suppose now that

$$\alpha(t) = te^{\omega}, \quad \beta(t) = te^{\eta}, \quad t \in \mathbb{R}_+,$$

$a_k, b_k \in \mathbb{C}$ for all $k \in \mathbb{Z}$ and

$$\sum_{k\in\mathbb{Z}} |a_k| < \infty, \quad \sum_{k\in\mathbb{Z}} |b_k| < \infty.$$

Then

$$\begin{aligned} N_{baby} &= \left(\sum_{k\in\mathbb{Z}} a_k U_\alpha^k\right) P_\gamma^+ + \left(\sum_{k\in\mathbb{Z}} b_k U_\beta^k\right) P_\gamma^- \\ &= \Phi^{-1}\,\mathrm{Co}(n_{baby})\Phi, \end{aligned}$$

where

$$n_{baby}(x) = \left(\sum_{k\in\mathbb{Z}} a_k e^{ik\omega x}\right) p_\gamma^+(x) + \left(\sum_{k\in\mathbb{Z}} b_k e^{ik\eta x}\right) p_\gamma^-(x), \quad x \in \mathbb{R}.$$

Fredholmness and invertibility of the operator N_{baby}

The function n_{baby} is a semi-almost periodic Fourier multiplier.

Theorem (after Sarason, 1977 and Duduchava-Saginashvili, 1981)

The following statements are equivalent:

- *the operator N_{baby} is Fredholm on the space $L^p(\mathbb{R}_+)$*
- *the operator N_{baby} is invertible on the space $L^p(\mathbb{R}_+)$*
- $$\inf_{x\in\mathbb{R}} |n_{baby}(x)| > 0.$$

Operator N and function n

Suppose

$$a_k, b_k \in SO(\mathbb{R}_+) \text{ for all } k \in \mathbb{Z}, \quad \alpha, \beta \in SOS(\mathbb{R}_+),$$

$$A_+ = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in W_{\alpha,p}^{SO}, \quad A_- = \sum_{k\in\mathbb{Z}} b_k U_\beta^k \in W_{\beta,p}^{SO}.$$

Since $\alpha(t) = te^{\omega(t)}$ and $\beta(t) = te^{\eta(t)}$ we can formally associate with the operator

$$N = N_{adult} = A_+ P_\gamma^+ + A_- P_\gamma^-$$

the function n as follows:

$$n(t,x) = n_{adult}(t,x) = \left(\sum_{k\in\mathbb{Z}} a_k(t) e^{ik\omega(t)x}\right) p_\gamma^+(x) + \left(\sum_{k\in\mathbb{Z}} b_k(t) e^{ik\eta(t)x}\right) p_\gamma^-(x),$$

$$(t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Operator N is not similar to a Mellin PDO

One might think, by analogy with

$$N_{baby} = \Phi^{-1}\,\mathrm{Co}(n_{baby})\Phi,$$

that

$$N_{adult} = \Phi^{-1}\,\mathrm{Op}(n_{adult})\Phi + \textit{compact operator}$$

where $\mathrm{Op}(a)$ is a Mellin PDO:

$$\mathrm{Op}(a)f(t) = [\mathcal{M}^{-1}a(t,\cdot)\mathcal{M}f](t), \quad t \in \mathbb{R}_+.$$

It is not the case!

Maximal ideal space of $C(\overline{\mathbb{R}}_+)$

For a unital commutative Banach algebra $\mathfrak{A}$, let $M(\mathfrak{A})$ denote its maximal ideal space.

Identifying the points $t \in \overline{\mathbb{R}}_+$ with the evaluation functionals

$$t(f) = f(t)$$

for $f \in C(\overline{\mathbb{R}}_+)$, we get

$$M(C(\overline{\mathbb{R}}_+)) = \overline{\mathbb{R}}_+.$$

Maximal ideal space of $SO(\mathbb{R}_+)$

Consider the fibers

$$M_s(SO(\mathbb{R}_+)) := \{\xi \in M(SO(\mathbb{R}_+)) : \xi|_{C(\overline{\mathbb{R}}_+)} = s\}$$

of the maximal ideal space $M(SO(\mathbb{R}_+))$ over the points $s \in \{0, \infty\}$.

The set

$$\Delta := M_0(SO(\mathbb{R}_+)) \cup M_\infty(SO(\mathbb{R}_+))$$

coincides with $(\mathrm{clos}_{SO^*} \mathbb{R}_+) \setminus \mathbb{R}_+$ where $\mathrm{clos}_{SO^*} \mathbb{R}_+$ is the weak-star closure of $\mathbb{R}_+$ in the dual space of $SO(\mathbb{R}_+)$. Then

$$M(SO(\mathbb{R}_+)) = \Delta \cup \mathbb{R}_+.$$

In what follows we write

$$a(\xi) := \xi(a) \quad \text{for} \quad a \in SO(\mathbb{R}_+), \quad \xi \in \Delta.$$

On the extension of function n to $M(SO(\mathbb{R}_+)) \times \mathbb{R}$

Under our assumption that

$$a_k, b_k \in SO(\mathbb{R}_+) \text{ for all } k \in \mathbb{Z}, \quad \alpha, \beta \in SOS(\mathbb{R}_+),$$

and

$$\sum_{k\in\mathbb{Z}} \|a_k\|_{C_b(\mathbb{R}_+)} < \infty, \quad \sum_{k\in\mathbb{Z}} \|b_k\|_{C_b(\mathbb{R}_+)} < \infty,$$

one can show that $n(\cdot, x) \in SO(\mathbb{R}_+)$ for every $x \in \mathbb{R}$.

Taking the Gelfand transform of $n(\cdot, x)$, we can extend the function $n(\cdot, x)$ defined on $\mathbb{R}_+$ to $M(SO(\mathbb{R}_+)) = \Delta \cup \mathbb{R}_+$, that is,

$$n(\xi, x) = \left(\sum_{k\in\mathbb{Z}} a_k(\xi) e^{ik\omega(\xi)x}\right) p_\gamma^+(x) + \left(\sum_{k\in\mathbb{Z}} b_k(\xi) e^{ik\eta(\xi)x}\right) p_\gamma^-(x)$$

for all $(\xi, x) \in (\Delta \cup \mathbb{R}_+) \times \mathbb{R}$.

Theorem (Main result: complete form)

Let $1 < p < \infty$ *and let* $\gamma \in \mathbb{C}$ *satisfy* $0 < 1/p + \Re\gamma < 1$. *Suppose*

$$a_k, b_k \in SO(\mathbb{R}_+) \text{ for all } k \in \mathbb{Z}, \quad \alpha, \beta \in SOS(\mathbb{R}_+),$$

$$A_+ = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in W_{\alpha,p}^{SO}, \quad A_- = \sum_{k\in\mathbb{Z}} b_k U_\beta^k \in W_{\beta,p}^{SO}.$$

For the operator

$$N = A_+ P_\gamma^+ + A_- P_\gamma^-,$$

the following assertions are equivalent:

(a) *the operator* N *is n-normal / d-normal on the space* $L^p(\mathbb{R}_+)$,

(b) *the operator* N *is left Fredholm / right Fredholm on* $L^p(\mathbb{R}_+)$,

(c) *the following two conditions are fulfilled:*

- (c-i) *the operators* A_+ *and* A_- *are left invertible / right invertible on the space* $L^p(\mathbb{R}_+)$;
- (c-ii) *for every* $\xi \in \Delta$, *the function* n *satisfies the inequality*

$$\inf_{x\in\mathbb{R}} |n(\xi, x)| > 0.$$

Why is the semi-Fredholm case much more difficult than the Fredholm case?

(a)⇒(c-i) Study of one-sided invertibility of A_+ and A_- is much more complicated than the study of their two-sided invertibility.

(a)⇒(c-ii) In the Fredholm case this implication can be obtained by using the method of limit operators, which is not applicable in the semi-Fredholm case. Instead we use a heavy machinery of Mellin pseudodifferental operators.

(c)⇒(b) One of the steps of the proof is to show that if

$$A_+ \in W^{SO}_{\alpha,p}, \quad A_- \in W^{SO}_{\beta,p}$$

are left invertible / right invertible then there are left inverses / right inverses $A_+^{(-1)}$ and $A_-^{(-1)}$ such that

$$A_+^{(-1)} \in W^{SO}_{\alpha,p}, \quad A_-^{(-1)} \in W^{SO}_{\beta,p}.$$

(b)⇒(a) trivial.

Thank you!