Moment Problems & Stable Polynomials

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THE TRUNCATED TRIGONOMETRIC MOMENT PROBLEM

FIX $d = (d_1, \ldots, d_n) \in \mathbb{N}^n$.

GIVEN $C_\alpha \in \mathbb{C}$ INDEXED BY

$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ WITH

$|\alpha_j| \leq d_j$.

WHEN ? $\exists$ POS MEASURE μ ON $\mathbb{T}^n$ S.T.

$$C_\alpha = \int_{\mathbb{T}^n} z^{-\alpha} \, d\mu \quad ?$$

$\boxed{\text{OPEN}}$

Truncated Trigonometric Moment Problem $n = 1$

<u>Thm</u>: Given $c_j \in \mathbb{C}$, $|j| \le d$.

$\exists$ a pos. measure μ with $c_j = \int_{\mathbb{T}} z^j \, d\mu$

if and only if

$$\left(c_{j-k} \right)_{j,k=0,\dots,n} \ge 0.$$

<u>Proof of</u> $\Rightarrow$ $\int \left| \sum_0^d a_j z^j \right|^2 d\mu = \sum a_j \bar{a}_k c_{j-k} \ge 0.$

STABLE POLYNOMIALS:

POLYNOMIALS WITH NO ZEROS IN "SOME REGION":

- POLYDISK $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$
- CLOSED POLYDISK $\overline{\mathbb{D}}^n$
- PRODUCT OF HALF-PLANES $\mathbb{C}_+^n$
- $\mathbb{C}_+^n \cup \mathbb{C}_-^n \implies$ "REAL STABLE"

STABLE POLYNOMIALS

MOTIVATION:

- FUNCTION THEORY, OPERATOR THEORY
- COMBINATORICS, PROBABILITY, STATISTICAL MECHANICS

(See work of Borcea-Brändén)

- OPTIMIZATION

Truncated Trigonometric Moment Problem
$n = 1$

Better Thm: Given $c_j \in \mathbb{C}$, $|j| \leq d$.

If $(c_{j-k})_{j,k=0,\dots,d} > 0$, then $\exists\, p \in \mathbb{C}[z]$ with $p(z) \neq 0$ for $z \in \overline{\mathbb{D}}$ = closed unit disk

such that

$$c_j = \int_{\mathbb{T}} z^j \frac{|dz|}{2\pi |p(z)|^2} \qquad |j| \leq d.$$

WHY IS THIS A BETTER THEOREM?

- SPECIFIC TYPE OF MEASURE AS SOLUTION.
- THE MEASURE SOLVES AN EXTREMAL PROBLEM: MAXIMIZES $\mathrm{dist}_{L^2(\nu)}(\mathcal{P}_d, z^{d+1})$ OVER SOLUTIONS ν.
- PROOF = BASIC HILBERT SPACE GEOMETRY (LITTLE FUNCTIONAL ANALYSIS)
- PROOF $\Rightarrow$ SUMS OF SQUARES FORMULA
- THIS THEOREM GENERALIZES TO 2 VARIABLES!

Sums of Squares Formula

Let $p \in \mathbb{C}[z]$, $p(z) \neq 0$ $z \in \overline{\mathbb{D}}$.

If $\deg p = d$, define $\tilde{p}(z) = z^d \overline{p(1/\bar{z})}$.

Then, $\exists A_1, \dots, A_d \in \mathbb{C}[z]$ such that

$$\overline{p(w)}\, p(z) - \overline{\tilde{p}(w)}\, \tilde{p}(z) = (1 - \bar{w} z) \sum_{j=1}^{d} \overline{A_j(w)}\, A_j(z)$$

GERONIMO-WOERDEMAN THM

2004 ANNALS OF MATH

GIVEN $c_\alpha \in \mathbb{C}$, $\alpha = (\alpha_1, \alpha_2)$, $|\alpha_j| \leq d_j$
$j = 1, 2$.

THERE EXISTS $p \in \mathbb{C}[z_1, z_2]$ WITH
$p(z) \neq 0$, $z \in \overline{\mathbb{D}}^2$ SUCH THAT

$$\boxed{c_\alpha = \int_{\mathbb{T}^2} z^\alpha \frac{|dz_1||dz_2|}{(2\pi)^2 |p(z)|^2}}, \quad \begin{array}{l} |\alpha_1| \leq d_1 \\ |\alpha_2| \leq d_2 \end{array}$$

IF AND ONLY IF $(c_{\alpha-\beta}) > 0$ AND

AN $\boxed{\text{EXTRA CONDITION}}$ HOLDS.

COMMENTS ON G-W THM

$\mu = \frac{1}{|p|^2} d\sigma$ IS CALLED A **BERNSTEIN-SZEGŐ** MEASURE

- EXTRA CONDITION = **ORTHOGONALITY RELATION** THAT HOLDS FOR $L^2(\mu)$.
- PROOF $\Rightarrow$ SUMS OF SQUARES FORMULA.

Sums of Squares Formula II

Let $p \in \mathbb{C}[z_1, z_2]$, $p(z) \neq 0$ $z \in \overline{\mathbb{D}}^2$.

If $\deg p = d = (d_1, d_2)$, then $\tilde{p}(z) := z_1^{d_1} z_2^{d_2} \overline{p\left(\frac{1}{\bar{z}_1}, \frac{1}{\bar{z}_2}\right)}$.

Then, $\exists\, A_1, \ldots, A_{d_1}, B_1, \ldots, B_{d_2} \in \mathbb{C}[z_1, z_2]$ such that

$$\overline{p(w)}\, p(z) - \overline{\tilde{p}(w)}\, \tilde{p}(z) = (1 - \bar{w}_1 z_1) \sum_{j=1}^{d_1} \overline{A_j(w)}\, A_j(z) + (1 - \bar{w}_2 z_2) \sum_{j=1}^{d_2} \overline{B_j(w)}\, B_j(z)$$

Comments on Sums of Squares Formula II

- "Squares" explicitly given as O.N. bases of subspaces of $L^2\left(\frac{1}{|p|^2}\,d\sigma\right)$.
- ((Cole-Wermer) implies Andô's inequality:

  $\|p(T_1, T_2)\| \le \|p\|_{\mathbb{D}^2}$, T_1, T_2 commuting contractions

  $p \in \mathbb{C}[z_1, z_2]$.
- Obvious generalization fails for $n > 2$.
- Implies transfer function realization for $f = \tilde{p}/p$.

TFRs & Determinantal Reps

(TFR) $p \in \mathbb{C}[z_1, z_2]$, $p(z) \neq 0$, $z \in \overline{\mathbb{D}}^2$, $\deg p = (d_1, d_2)$

$$\frac{\tilde{p}}{p}(z) = A + B\begin{pmatrix} z_1 I_{d_1} & \\ & z_2 I_{d_2} \end{pmatrix}\left(I - D\begin{pmatrix} z_1 I_{d_1} & \\ & z_2 I_{d_2} \end{pmatrix}\right)^{-1} C$$

$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ unitary matrix

(Det Rep) $\boxed{p(z) = c \det\left(I - D\begin{pmatrix} z_1 I_{d_1} & \\ & z_2 d_2 \end{pmatrix}\right)}$

(Det Rep) $q \in \mathbb{C}[z_1, z_2]$, $q(z) \neq 0$ $z \in UHP^2$

$$q(z) = \boxed{c \det(A + z_1 B_1 + z_2 B_2)}$$

$\operatorname{Im} A \geq 0$, $B_1, B_2 \geq 0$, $I = B_1 + B_2$

CAN WE GENERALIZE FURTHER?

① STUDY $p \in \mathbb{C}[z_1, z_2]$ WITH NO ZEROS IN $\mathbb{D}^2$ AND FINITELY MANY IN $\mathbb{T}^2$

(See Knese APDE 2009, Knese PLMS 2015)

② STUDY $p \in \mathbb{C}[z_1, z_2]$ WITH NO ZEROS IN $\mathbb{T} \times \overline{\mathbb{D}}$

(See Geronimo-Iliev-Knese JFA 2016)

$p \in \mathbb{C}[z_1, z_2]$ WITH NO ZEROS IN $\mathbb{D}^2$
AND FINITELY MANY IN $\mathbb{T}^2$

EXAMPLE: $p(z) = 2 - z_1 - z_2$

WHY? • LEARN ABOUT EFFECT OF BOUNDARY ZEROS

WHY "FINITELY MANY"? $p \in \mathbb{C}[z_1, z_2]$, $p(z) \neq 0$ $\mathbb{D}^2$

BREAK INTO 2 CASES: $p = c\tilde{p}$, $\gcd(p, \tilde{p}) = 1$.

- $p = c\tilde{p} \Rightarrow Z_p \cap \mathbb{T}^2 = \text{CURVE}$ **EX:** $1 - z_1 z_2$
- $\gcd(p, \tilde{p}) = 1 \Rightarrow Z_p \cap \mathbb{T}^2$ FINITE.

WHAT HAPPENS?

Moment Problems (or lack thereof) & Stable Polynomials

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$p \in \mathbb{C}[z_1, z_2]$ WITH NO ZEROS IN $\mathbb{D}^2$
AND $\gcd(p, \tilde{p}) = 1$

WHAT HAPPENS?

- ZEROS ON $\mathbb{T}^2 \implies$ MOMENTS $\int_{\mathbb{T}^2} z^{\alpha} \frac{1}{|p|^2} d\sigma$ NO LONGER EXIST!
- MUST LOOK AT IDEAL:

$$\mathcal{J}_p = \left\{ q \in \mathbb{C}[z_1, z_2] : \int_{\mathbb{T}^2} \left| \frac{q}{p} \right|^2 d\sigma < \infty \right\}$$

- CAN CONSTRUCT GENERATORS USING SUMS OF SQUARES FORMULA
- $\operatorname{Codim} \mathcal{J}_p = \frac{1}{2} \# Z_p \cap Z_{\tilde{p}} \cap \mathbb{T}^2$ COUNT CORRECTLY!

$p \in \mathbb{C}[z_1, z_2]$ WITH NO ZEROS IN $\mathbb{D}^2$
AND $\gcd(p, \tilde{p}) = 1$

WHAT HAPPENS?

- CAN COUNT THE NUMBER OF DISTINCT SUMS OF SQUARES FORMULAS:

$$|p|^2 - |\tilde{p}|^2 = \sum_{j=1}^{2} (1 - |z_j|^2)\, SOS_j$$

USING INVARIANT SUBSPACES OF

$$(T_1, T_2^*)$$

$$K = \{ q \in \mathcal{J}_p : \deg q \le (d_1 - 1, d_2 - 1) \}$$

IN $L^2\left(\frac{1}{|p|^2} d\sigma\right)$

$$T_j = P_K M_{z_j}\big|_K$$

$p \in \mathbb{C}[z_1, z_2]$ WITH NO ZEROS IN $\mathbb{T} \times \overline{\mathbb{D}}$

WHY?

- THE THEORY WORKS!
  - SUMS OF SQUARES FORMULA (NEXT SLIDE)
  - "EXTRA CONDITION" IN GERONIMO–WOERDEMAN MORE COMPLICATED
- $p \in \mathbb{C}[z_1, z_2]$ WITH NO ZEROS IN $\mathbb{T} \times \overline{\mathbb{D}}$

$\xRightarrow{\text{CAYLEY TRANS}}$ $q \in \mathbb{C}[z_1, z_2]$ WITH NO ZEROS IN $\mathbb{R} \times \overline{UHP}$

(CLOSELY RELATED TO <u>HYPERBOLIC</u> <u>POLYNOMIALS</u>)

$p \in \mathbb{C}[z_1, z_2]$ WITH NO ZEROS IN $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$

SUMS OF (DIFFERENCE OF) SQUARES:

$$|p(z)|^2 - |\tilde{p}(z)|^2 = (1-|z_1|^2)\left[\sum_{j=1}^{n_1} |A_j(z)|^2 - \sum_{j=1}^{n_2} |B_j(z)|^2\right]$$

$$+ (1-|z_2|^2)\sum_{j=1}^{d_2} |C_j(z)|^2$$

$d_1 = n_1 + n_2$, n_2 = # zeros of $p(z_1, 0)$ in $\mathbb{D}$

- IMPLIES TRANSFER FUNCTION REALIZATION, DET. REP.
- DOES IT IMPLY SOME ANDÔ INEQUALITY?

THE FUTURE

- STUDY $p \in \mathbb{C}[z_1, z_2]$ WITH NO ZEROS IN $\mathbb{T} \times \mathbb{D}$ AND $\gcd(p, \tilde{p}) = 1$.
- CAN WE STUDY MORE GENERAL p?
- THREE OR MORE VARIABLES?
  - $p \in \mathbb{C}[z_1, \dots z_n]$ WITH SOS FORMULA
  - $p \in \mathbb{C}[z_1, \dots, z_n]$ WITHOUT

SUMMARY

- THE "RIGHT" APPROACH TO 1-VAR MOMENT PROBLEM

$\Rightarrow$ A THEOREM IN 2-VAR

$\Rightarrow$ DETAILED INFORMATION ABOUT STABLE POLYNOMIALS

DAS ENDE

DANKE!

"Basic" Hilbert Space Geometry

Fix $d \in \mathbb{N}^n$. Let $c_\alpha \in \mathbb{C}$, $-d \le \alpha \le d$

Assume $\left(c_{\alpha-\beta}\right)_{0 \le \alpha, \beta \le d} > 0$. $(*)$ $\quad \alpha \in \mathbb{Z}^n$.

Let $\mathcal{L}_d = \operatorname{span}\{ z^\alpha : -d \le \alpha \le d \}$

$\mathcal{P}_d = \operatorname{span}\{ z^\alpha : \underline{0} \le \alpha \le d \}$.

Define $L : \mathcal{L}_d \to \mathbb{C}$, $\quad L(z^\alpha) = c_{-\alpha}$

$$\langle f, g \rangle_L = L\left(f(z) \overline{g(1/\bar{z})} \right) \quad f, g \in \mathcal{P}_d .$$

An inner product by $(*)$

$\underline{n=1}$

$$\boxed{\langle f, g\rangle_L = L\left(f(z)\overline{g(1/\bar{z})}\right)}$$

- LET $p \in \mathcal{P}_d \ominus z\,\mathcal{P}_{d-1}$, $\|p\|_L = 1$.
- Then $\tilde{p}(z) := z^d\,\overline{p(1/\bar{z})} \in \mathcal{P}_d \ominus \mathcal{P}_{d-1}$.
- LET $A_0, \ldots, A_{d-1}$ be an o.n. basis for $\mathcal{P}_{d-1}$.
- Then, $\{p, zA_0, \ldots, zA_{d-1}\}$, $\{\tilde{p}, A_0, \ldots, A_{d-1}\}$

form o.n. bases for $\mathcal{P}_d$.

- $\exists$ unitary $\mathcal{U}$ such that

$$\mathcal{U}\begin{pmatrix} p \\ zA_0 \\ \vdots \\ zA_{d-1} \end{pmatrix} = \begin{pmatrix} \tilde{p} \\ A_0 \\ \vdots \\ A_{d-1} \end{pmatrix}$$

$$\boxed{\begin{array}{l} U\begin{pmatrix} p \\ z A_0 \\ \vdots \\ z A_{d-1} \end{pmatrix} = \begin{pmatrix} \tilde{p} \\ A_0 \\ \vdots \\ A_{d-1} \end{pmatrix} \\ p \in \mathcal{P}_d \ominus z\,\mathcal{P}_{d-1} \end{array}}$$

$$\Rightarrow \overline{p(w)}\,p(z) - \overline{\tilde{p}(w)}\,\tilde{p}(z) = (1-\bar{w}z)\sum_{j=0}^{d-1} \overline{A_j(w)}\,A_j(z)$$

$$\Rightarrow p(z) \neq 0, \quad z \in \overline{\mathbb{D}}$$

$$\Rightarrow p \in \mathcal{P}_d \ominus z\,\mathcal{P}_{d-1} \text{ determines } \langle\, , \rangle_L$$

- Now, for $\mu = \frac{1}{|p|^2} d\sigma$, $p \in \mathcal{P}_d \ominus z\,\mathcal{P}_{d-1}$ BY

$$\int_{\mathbb{T}} z^j\,\bar{p}\,\frac{1}{|p|^2}\frac{|dz|}{2\pi} = \begin{cases} 0 & j>0 \\ \frac{1}{p(0)} & j=0. \end{cases}$$