Spectral properties of approximation sequences:

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Outline of the Talk

1. Finite section method
   - Stability
2. Algebras of operator sequences
   - Convergence notions
   - Algebras of T-structured sequences
   - Fractal algebras
3. Rich sequences
   - T-structured subsequences
   - Passage from sequences to subsequences
4. Finite sections of convolution type operators
   - Multiplication and convolution operators
   - An algebra $\mathscr{A}$ of convolution type operators
   - Approximation sequences to operators in $\mathscr{A}$

Outline of the Talk

Approximate solution to an operator equation

Let $A \in \mathscr{L}(L^p(\mathbb{R}))$, with $1 < p < \infty$, and an approximate sequence $A_n \in \mathscr{L}((L^p(\mathbb{R}))$ of A, based on a sequence of projections $P_n \in \mathscr{L}(L^p(\mathbb{R}))$. A common question is to know whether we can substitute the equation

$$Au = b, \quad u, b \in L^p(\mathbb{R})$$

by the "simpler" ones

$$A_n u_n = P_n b$$

and guarantee that u_n are unique and converge to the solution of the initial equation.

- **Stability**: The sequence (A_n) is stable if for n large enough the operators A_n are invertible and $\sup \|A_n^{-1}\| < \infty$.

Outline of the Talk

Convergence notions

Let $\mathscr{P} = (P_n) := \chi_{[-n,n]} I$.

- $\mathscr{P}$-compact operators

$$\mathscr{K}(L^p, \mathscr{P}) := \{K \in \mathscr{L}(L^p(\mathbb{R})) : \|K(I - P_n)\|, \|(I - P_n)K\| \to 0 \text{ as } n \to \infty\}.$$

$$\mathscr{L}(L^p, \mathscr{P}) := \{A \in \mathscr{L}(L^p(\mathbb{R})) : AK, KA \in \mathscr{K}(L^p, \mathscr{P}), \ \forall K \in \mathscr{K}(L^p, \mathscr{P})\}$$

- $\mathscr{P}$-Fredholm operators
  $A \in \mathscr{L}(L^p, \mathscr{P})$ is $\mathscr{P}$-Fredholm if $A + \mathscr{K}(L^p, \mathscr{P})$ is invertible in the quotient algebra $\mathscr{L}(L^p, \mathscr{P}) / \mathscr{K}(L^p, \mathscr{P})$.
- $\mathscr{P}$-Convergence: A sequence $(A_n) \subset \mathscr{L}(L^p, \mathscr{P})$ is said to **converge** **$\mathscr{P}$-strongly** to $A \in \mathscr{L}(L^p(\mathbb{R}))$ if

$$\|K(A_n - A)\|, \|(A_n - A)K\| \to 0 \text{ as } n \to \infty$$

  for every $\mathscr{P}$-compact operators K.
- A sequence $(A_n) \subset \mathscr{L}(L^p(\mathbb{R}))$ is said to **converge** **$*$-strongly** to $A \in \mathscr{L}(L^p(\mathbb{R}))$ if

$$s\text{-}\lim_{n\to\infty} A_n = A \qquad \text{and} \qquad s\text{-}\lim_{n\to\infty} A_n^* = A^*$$

Outline of the Talk

T-structured sequences

The set $\mathscr{F}$ defined by

$$\mathscr{F} := \{(A_n) : A_n \in \mathscr{L}(L^p, \mathscr{P}) \text{ and } \sup_n \|A_n\| < \infty\}$$

is a Banach algebra. Consider the following 3 homomorphisms on $L^p(\mathbb{R})$:
$(V_n u)(x) := u(x-n)$,
$(Z_n u)(x) := n^{-1/p} u(x/n)$ and
$(U_t u)(x) := e^{itx} u(x)$, $t \in \mathbb{R}$.
Let $\mathscr{F}^T$ be the set of all T-**structured sequences**, i.e. all sequences $\mathbb{A} = (A_n) \in \mathscr{F}$ for which the $\mathscr{P}$-strong limits

$$\mathrm{W}(\mathbb{A}) := \mathscr{P}\text{-}\lim_{n\to\infty} A_n, \quad \mathrm{W}^{\pm}(\mathbb{A}) := \mathscr{P}\text{-}\lim_{n\to\infty} V_{\mp n} A_n V_{\pm n},$$

exist and the $*$-strong limits

$$\mathrm{H}^t(\mathbb{A}) := s\text{-}\lim_{n\to\infty} Z_n^{-1} U_t A_n U_t^{-1} Z_n$$

also exist for every $t \in \mathbb{R}$, where $T := \left\{(V_{\mp n}), \left(Z_n^{-1} U_t\right), t \in \mathbb{R}\right\}$.

This set forms a closed subalgebra of $\mathscr{F}$.

$\mathscr{J}^T$-Fredholm sequences

$\mathscr{F}^T$ contains the ideal

$$\mathscr{J}^T := \{(K) + (V_{\pm n} K^- V_{\mp}) + (U_t^{-1} Z_n K^+ Z_n^{-1} U_t) + (G_n) : \\ K, K^- \in \mathscr{K}(X, \mathscr{P}),\ K^+ \in \mathscr{K} \text{ and } \|G_n\| \to 0 \text{ as } n \to \infty\}$$

and we say that $(A_n) \in \mathscr{F}^T$ is a $\mathscr{J}^T$-**Fredholm sequence** if $(A_n) + \mathscr{J}^T$ is invertible in $\mathscr{F}^T / \mathscr{J}^T$.

The following theorem is an adaptation of the Silbermann´s lifting theorem

Theorem

Let $\mathbb{A} = (A_n) \in \mathscr{F}^T$. Then $\mathbb{A}$ is stable if and only if $\mathbb{A}$ is $\mathscr{J}^T$-Fredholm and $\mathrm{W}(\mathbb{A})$, $\mathrm{W}^+(\mathbb{A})$, $\mathrm{W}^-(\mathbb{A})$ and $\mathrm{H}^t(\mathbb{A})$, $\forall t \in \mathbb{R}$, are invertible.

Outline of the Talk

Definition

Let $\mathscr{B}$ be a Banach subalgebra of $\mathscr{F}$ containing $\mathscr{G} := \{(G_n) : \|G_n\| \to 0 \text{ as } n \to \infty\}$.
$\mathscr{B}$ is a **fractal algebra** if for every strictly increasing sequence h of natural numbers and $\mathscr{B}_h := \{(A_{h_n}) : (A_n) \in \mathscr{B}\}$, there exists a map $\pi_h : \mathscr{B}_h \to \mathscr{B}/\mathscr{G}$ such that for every $\mathbb{A} \in \mathscr{B}$,

$$\mathbb{A} + \mathscr{G} = \pi_h(\mathbb{A}_h).$$

Theorem

[Roch, Silbermann 1996] Let $p = 2$. If $\mathscr{B}$ is a unital fractal subalgebra of $\mathscr{F}$ and $\mathbb{A} = (A_n) \in \mathscr{B}$ then,

- *$\mathbb{A}$ is stable if and only if it possesses a stable subsequence.*
- *The limit $\lim \|A_n\|$ exists and equals $\|\mathbb{A} + \mathscr{G}\|$.*

Outline of the Talk

T-structured subsequences

Given a strictly increasing sequence h of natural numbers and the associated projections $P_{h_n} = \chi_{[-h_n, h_n]} I$ we define in analogy the sets

$$\mathscr{F}_h = \{(A_{h_n}) : (A_n) \in \mathscr{F}\},$$

$\mathscr{F}_h^T$, $\mathscr{J}_h^T$, and denote by $\mathrm{W}(\mathbb{A}_h)$, the limit operators of the subsequences $\mathbb{A}_h := \{A_{h_n}\}$.

Definition

A sequence $\mathbb{A} = (A_n) \in \mathscr{F}$ is **rich** if every subsequence of $\mathbb{A}$ has a T-structured subsequence $\mathbb{A}_h = (A_{h_n})$, i.e. for every strictly increasing sequence g of natural numbers there exists a subsequence h such that $\mathbb{A}_h \in \mathscr{F}_h^T$.

- $\mathscr{R}^T$ denote the subset of $\mathscr{F}$ consisting of all rich sequences.

$$\mathscr{F}^T \subset \mathscr{R}^T \subset \mathscr{F}$$

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Passage from sequences to subsequences

Seidel, Silbermann, 2012

A sequence is $\mathbb{A} \in \mathscr{R}^T$ is stable if and only if every T-structured subsequence $\mathbb{A}_h$ is stable.

As a consequence of the previous theorem and the lifting theorem:

Let $\mathbb{A} \in \mathscr{R}^T$. Then, $\mathbb{A}$ is stable if and only if every T-structured subsequence $\mathbb{A}_h$ has a $\mathscr{J}^T$-Fredholm subsequence and $\mathrm{W}(\mathbb{A}_h)$, $\mathrm{W}^+(\mathbb{A}_h)$, $\mathrm{W}^-(\mathbb{A}_h)$ and $\mathrm{H}^t(\mathbb{A})$, $\forall t \in \mathbb{R}$, are invertible.

Outline of the Talk

Function algebras

Closed subalgebras of $L^\infty(\mathbb{R})$:

- BUC - the algebra of bounded and uniformly continuous functions on $\mathbb{R}$
- L_0^∞ - the algebra of functions with which possesses finite limit at $\pm\infty$.
- PC^λ - the algebra of continuous functions with one-sided limits at $\lambda \in \mathbb{R}$
- PC - the algebra generated by all PC^λ with $\lambda \in \mathbb{R}$.
- SO^λ - the algebra of continuous functions on $\dot{\mathbb{R}} \setminus \{\lambda\}$ and slowly oscillating at $\lambda \in \dot{\mathbb{R}}$, i.e.

$$\lim_{x\to+0} \operatorname{osc}(f, \lambda + ([-x,-rx] \cup [rx,x])) = 0 \quad \text{if} \quad \lambda \in \mathbb{R}$$

$$\lim_{x\to+\infty} \operatorname{osc}(f, [-x,-rx] \cup [rx,x]) = 0 \quad \text{if} \quad \lambda = \infty$$

for every $r \in \,]0,1[$, where $\operatorname{osc}(f,I) := \operatorname{esssup}\{|f(t)-f(s)| : t,s \in I\}$.
- SO - the algebra generated by all SO^λ with $\lambda \in \mathbb{R}$.

Convolution operators:

For $b \in L^\infty(\mathbb{R})$ and F being the Fourier transform, the *convolution operator* on $L^2(\mathbb{R})$, is defined by:

$$W^0(b) := F^{-1} bF ,$$

If the operator $W^0(b)$ on $L^p(\mathbb{R}) \cap L^2(\mathbb{R})$ admits a linear bounded extension to $L^p(\mathbb{R})$ then it is also called a convolution operator and b is called a *Fourier multiplier*. The set of all multipliers on $L^p(\mathbb{R})$

$$M^p := \{b \in L^\infty(\mathbb{R}) : W^0(b) \in \mathscr{L}(L^p(\mathbb{R}))\}$$

with the norm $\|b\|_{M^p} := \|W^0(b)\|_{\mathscr{L}(L^p(\mathbb{R}))}$ forms a Banach algebra. For $p \in (1,\infty) \setminus \{2\}$, let $M^{<p>}$ denote the set of all multipliers $b \in M^p$ for which there exists a $\delta > 0$ (depending on b) such that $b \in M^r$ for all $r \in (p-\delta, p+\delta)$. Also set $M^{<2>} := M^2 = L^\infty(\mathbb{R})$. Furthermore, for a subalgebra $\mathscr{B} \subset L^\infty(\mathbb{R})$ let $\mathscr{B}_p$ denote the closure in M^p of $\mathscr{B} \cap M^{<p>}$.

Outline of the Talk

An algebra $\mathscr{A}$ of convolution type operators

Let $\mathscr{A}$ be the smallest closed subalgebra of $\mathscr{L}(L^p(\mathbb{R}))$ which contains:

- All operators of multiplication aI, with $a \in \text{alg}(L_0^\infty, PC, SO)$
- All convolution operators $W^0(b)$ with $b \in [\text{alg}(BUC, PC^\lambda, SO)]_p$ $\forall \lambda \in \mathbb{R}$
- All $\mathscr{P}$-compact operators

Let $\mathscr{F}_{\mathscr{A}}$ denote the smallest closed subalgebra of $\mathscr{F}$ containing all finite sections

$$(P_n A P_n + (I - P_n)), \quad A \in \mathscr{A}.$$

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Approximation sequences to operators in $\mathscr{A}$

Theorem

Let $\mathbb{A} \in \mathscr{F}_{\mathscr{A}}$. $\mathbb{A}$ is stable if and only if for every T-structured subsequence $\mathbb{A}_h$, the operators $\mathrm{W}(\mathbb{A}_h)$, $\mathrm{W}^+(\mathbb{A}_h)$, $\mathrm{W}^-(\mathbb{A}_h)$ and $\mathrm{H}^t(\mathbb{A}_h)$, $\forall t \in \mathbb{R}$, are invertible.

Remarks: For $p = 2$
(1) Every T-structured subsequence belongs to a fractal algebra
(2) Results on the index, asymptotic behaviour of condition numbers and convergence of pseudospectrum are also obtained for $\mathscr{F}_{\mathscr{A}}$.

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