Representation Theorems for Solvable Sesquilinear Forms

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1. S. Di Bella, C. Trapani, *Some representation theorems for sesquilinear forms*, J. Math. Anal. Appl., 451 (2017), 64-83.
2. R. Corso, C. Trapani, *Representation Theorems for Solvable Sesquilinear Forms*, Integral Equations Operator Theory, published online, 2017.
3. R. Corso, *A Kato's second type representation theorem for solvable sesquilinear forms*, arXiv:1707.05073, math.FA, 2017.

Let $\mathcal{H}$ be a Hilbert space, with inner product $\langle\cdot|\cdot\rangle$ and norm $||\cdot||$.

Theorem

If Ω is a bounded sesquilinear form on $\mathcal{H}$, then there exists a unique bounded operator T such that

$$\Omega(\xi,\eta) = \langle T\xi|\eta\rangle \qquad \forall \xi,\eta \in \mathcal{H}.$$

Theorem (Kato's first representation theorem)

Let Ω be a closed, sectorial sesquilinear form on a dense subspace $\mathcal{D}$ of $\mathcal{H}$. Then, there exists a unique m-sectorial operator T, with $D(T) \subseteq \mathcal{D}$, such that

$$\Omega(\xi,\eta) = \langle T\xi|\eta\rangle, \qquad \forall \xi \in D(T), \eta \in \mathcal{D}.$$

Q-closed and solvable sesquilinear forms

Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$.

Definition

A sesquilinear form Ω on $\mathcal{D}$ is called *q-closed with respect to* a norm $||\cdot||_\Omega$ on $\mathcal{D}$ if

1. $\mathcal{E}_\Omega := \mathcal{D}[||\cdot||_\Omega]$ is a reflexive Banach space;
2. the embedding $\mathcal{E}_\Omega \to \mathcal{H}$ is continuous;
3. Ω is bounded on $\mathcal{E}_\Omega$.

Under these hypotheses, a *Banach-Gelfand triplet* $\mathcal{E}_\Omega \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{E}_\Omega^\times$ is well-defined.

Let Ω be a q-closed sesquilinear form with respect to a norm $||\cdot||_\Omega$ on $\mathcal{D}$. Let Υ be a bounded sesquilinear form on $\mathcal{H}$. We can define the bounded operator

$$\begin{aligned} X_\Upsilon : \mathcal{E}_\Omega &\to \mathcal{E}_\Omega^\times \\ \xi &\mapsto \Omega_\Upsilon^\xi, \end{aligned}$$

where $\langle \Omega_\Upsilon^\xi | \eta \rangle = \Omega(\xi, \eta) + \Upsilon(\xi, \eta)$, for all $\eta \in \mathcal{E}_\Omega$.
We denote by $\mathfrak{P}(\Omega)$ the set of bounded sesquilinear forms Υ on $\mathcal{H}$, such that X_Υ is a bijection.

Definition

If the set $\mathfrak{P}(\Omega)$ is not empty, then Ω is said to be *solvable with respect to* $||\cdot||_\Omega$.

The first representation theorem

Theorem

Let Ω be a solvable sesquilinear form on $\mathcal{D}$ with respect to a norm $||\cdot||_\Omega$. Then there exists a closed operator T, with dense domain $D(T) \subseteq \mathcal{D}$ in $\mathcal{H}$, such that

$$\Omega(\xi,\eta) = \langle T\xi|\eta\rangle, \qquad \forall \xi \in D(T), \eta \in \mathcal{D}.$$

The operator T is said *associated* to Ω.

The construction of T

By hypothesis $\mathfrak{P}(\Omega) \neq \varnothing$. Let $\Upsilon \in \mathfrak{P}(\Omega)$, then $X_\Upsilon : \mathcal{E}_\Omega \to \mathcal{E}_\Omega^\times$ is a bijection.

$$\begin{array}{rcc} \mathcal{E}_\Omega & \xrightarrow{X_\Upsilon} & \mathcal{E}_\Omega^\times \\ \cup & & \cup \\ D(T) := X_\Upsilon{}^{-1}\mathcal{H} & \xrightarrow{X_\Upsilon} & \mathcal{H} \end{array}$$

$$T\xi := X_\Upsilon \xi - B\xi, \qquad \forall \xi \in D(T),$$

where $B \in \mathcal{B}(\mathcal{H})$ is such that $\Upsilon(\cdot,\cdot) = \langle B \cdot | \cdot \rangle$.

T does not depend on the choice of $\Upsilon \in \mathfrak{P}(\Omega)$.

Some properties of T

1. $D(T)$ is dense in $\mathcal{E}_\Omega := \mathcal{D}[||\cdot||_\Omega]$.
2. If T' is an operator with $D(T') \subseteq \mathcal{D}$ and $\Omega(\xi,\eta) = \langle T'\xi|\eta\rangle$, $\forall \xi \in D(T')$ and η which belongs to a dense subset of $\mathcal{E}_\Omega$, then $T' \subseteq T$.
3. A bdd form $\Upsilon(\cdot,\cdot) = \langle B\cdot|\cdot\rangle \in \mathfrak{P}(\Omega) \Leftrightarrow 0 \in \rho(T+B)$.
   If $\Upsilon(\cdot,\cdot) = -\lambda\langle\cdot|\cdot\rangle$, with $\lambda \in \mathbb{C}$, $\Upsilon \in \mathfrak{P}(\Omega) \Leftrightarrow \lambda \in \rho(T)$.
4. if $\Upsilon(\cdot,\cdot) = \langle B\cdot|\cdot\rangle \in \mathfrak{P}(\Omega)$ then T is the unique operator S satisfying

$$\Omega(\xi,\eta) = \langle S\xi|\eta\rangle, \qquad \forall \xi \in D(S), \eta \in \mathcal{D}$$

   and such that $S+B$ has range $\mathcal{H}$.

Theorem

Let Ω be a q-closed (respectively solvable) sesquilinear form on $\mathcal{D}$ with respect to a norm $||\cdot||_1$ and let $||\cdot||_2$ be a norm on $\mathcal{D}$. Then, Ω is q-closed (respectively solvable) with respect to $||\cdot||_2$ if, and only if, $||\cdot||_1$ and $||\cdot||_2$ are equivalent.

Proposition

Let Ω be a q-closed sesquilinear form on $\mathcal{D}$ with respect to a norm $||\cdot||_\Omega$ with numerical range $\mathfrak{n}_\Omega := \{\Omega(\xi,\xi) : \xi \in \mathcal{D}, ||\xi|| = 1\} \neq \mathbb{C}$. If $\lambda \notin \mathfrak{n}_\Omega$, then $-\lambda\langle\cdot|\cdot\rangle \in \mathfrak{P}(\Omega)$ if, and only if, there exists a constant $c > 0$ such that

$$c||\xi||_\Omega \leq \sup_{||\eta||_\Omega=1} |(\Omega - \lambda\iota)(\xi,\eta)|, \qquad \forall \xi \in \mathcal{D}.$$

Theorem

If Ω is a q-closed (respectively solvable) sesquilinear form on $\mathcal{D}$ with respect to a norm $||\cdot||_\Omega$, then also the adjoint Ω^* is q-closed (respectively solvable) with respect to $||\cdot||_\Omega$.
If Ω is solvable and T is its associated operator, then T^* is the operator associated to Ω^*.

Corollary

The operator associated to a solvable sesquilinear form is self-adjoint if, and only if, the form is symmetric.

Special cases

The class of solvable forms covers the sesquilinear forms considered by many authors, for instance:

- Lions (1961)
- Kato (1966)
- McIntosh (1968, 1970)
- Fleige, Hassi, de Snoo (2000)
- Schmüdgen (2012)
- Grubišić, Kostrykin, Makarov, Veselić (2013)
- Schmitz (2015).

Example

Let Ω be the sesquilinear form with domain

$$\mathcal{D} := \left\{ f \in L^2(\mathbb{C}) : \int_{\mathbb{C}} |z||f(z)|^2 dz < \infty \right\}$$

given by $\Omega(f,g) = \displaystyle\int_{\mathbb{C}} zf(z)\overline{g(z)}dz$, $f, g \in \mathcal{D}$.
Ω is solvable with respect to the norm

$$||f||_\Omega = \left(\int_{\mathbb{C}} (1+|z|)|f(z)|^2 dz \right)^{\frac{1}{2}}, \qquad f \in \mathcal{D}.$$

Ω does not satisfy the conditions of other representation theorems (in particular Ω is not sectorial). The operator associated to Ω is the multiplication operator by z on $L^2(\mathbb{C})$.

The second representation theorem

Theorem (Kato's second representation theorem)

Let Ω be closed, positive sesquilinear form on $\mathcal{D}$ and let T be the positive self-adjoint associated operator. Then $\mathcal{D} = D(T^{\frac{1}{2}})$ and

$$\Omega(\xi, \eta) = \langle T^{\frac{1}{2}}\xi | T^{\frac{1}{2}}\eta \rangle, \qquad \forall \xi, \eta \in \mathcal{D}.$$

Definition

A solvable sesquilinear form on $\mathcal{D}$ with associated operator T is said *hyper-solvable* if $\mathcal{D} = D(|T|^{\frac{1}{2}})$.

Example

The multiplication form by z on $L^2(\mathbb{C})$ is hyper-solvable.

Lemma

If Ω is a hyper-solvable sesquilinear form on $\mathcal{D}$ with associated operator T, then $\mathcal{D} = D(|T^*|^{\frac{1}{2}})$.

Theorem

Let Ω be a hyper-solvable sesquilinear form on $\mathcal{D}$ with respect to a norm $||\cdot||_\Omega$ and with associated operator T. Then

$$\Omega(\xi, \eta) = \langle U|T|^{\frac{1}{2}}\xi | |T^*|^{\frac{1}{2}}\eta\rangle, \qquad \forall \xi, \eta \in \mathcal{D},$$

$$\Omega(\xi, \eta) = \langle |T^*|^{\frac{1}{2}}U\xi | |T^*|^{\frac{1}{2}}\eta\rangle, \qquad \forall \xi, \eta \in \mathcal{D},$$

where $T = U|T| = |T^*|U$ is the polar decomposition of T, and $||\cdot||_\Omega$ is equivalent to the graph norms of $|T|^{\frac{1}{2}}$ and of $|T^*|^{\frac{1}{2}}$.

Example

Let Ω be the multiplication form by z on $L^2(\mathbb{C})$, $\mathcal{D}$ be its domain and M be its associated operator.

$$\Omega(\xi,\eta) = \langle U|M|^{\frac{1}{2}}\xi | |M|^{\frac{1}{2}}\eta\rangle, \qquad \forall \xi,\eta \in \mathcal{D},$$

$$\Omega(\xi,\eta) = \langle |M|^{\frac{1}{2}}U\xi | |M|^{\frac{1}{2}}\eta\rangle, \qquad \forall \xi,\eta \in \mathcal{D},$$

where $M = U|M|$ is the polar decomposition of M and in addition $|M| = |M^*|$.

More precisely, $|M|^{\frac{1}{2}}$ and U are the multiplication operators by, respectively, $|z|^{\frac{1}{2}}$ and $s(z)$, where $s(z) = \frac{z}{|z|}$ if $z \neq 0$ and $s(0) = 0$.

Theorem

Let T be a densely defined, closed operator satisfies

(a) *there exists $B \in \mathcal{B}(\mathcal{H})$ such that $0 \in \rho(T + B)$;*

(b) $D(|T|^{\frac{1}{2}}) = D(|T^*|^{\frac{1}{2}})$.

Then, there exists a unique hyper-solvable sesquilinear form Ω with associated operator T.

$$\{\text{hyper-solvable forms}\} \stackrel{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{densely defined closed} \\ \text{operators verifying (a) and (b)} \end{array} \right\}$$

$$\{\text{symmetric hyper-solvable forms}\} \stackrel{1-1}{\longleftrightarrow} \{\text{self-adjoint operators}\}$$

Proposition

Let Ω_1 and Ω_2 be two solvable sesquilinear forms with domains $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, and with the same associated operator T. If $\mathcal{D}_1 \subseteq \mathcal{D}_2$ then $\mathcal{D}_1 = \mathcal{D}_2$ and $\Omega_1 = \Omega_2$.

Proposition

Let Ω be a solvable sesquilinear form on $\mathcal{D}$, with associated operator T to Ω. The following statements are equivalent.

1. $\mathcal{D} = D(|T|^{\frac{1}{2}})$, *i.e.* Ω *is hyper-solvable;*
2. $\mathcal{D} \subseteq D(|T|^{\frac{1}{2}}) \cap D(|T^*|^{\frac{1}{2}})$;
3. $\mathcal{D} \supseteq D(|T|^{\frac{1}{2}}) \cup D(|T^*|^{\frac{1}{2}})$.

Theorem

A sesquilinear form Ω *on* $\mathcal{D}$ *is q-closed with respect to a norm induced by an inner product, if and only if, there exist a positive self-adjoint operator* H*, with domain* $D(H) = \mathcal{D}$ *and* $0 \in \rho(H)$*, and* $Q \in \mathcal{B}(\mathcal{H})$ *such that*

$$\Omega(\xi, \eta) = \langle QH\xi | H\eta \rangle, \qquad \forall \xi, \eta \in \mathcal{D}. \tag{1}$$

Suppose that (1) holds and that Ω *is also solvable. Then its associated operator is* $T = HQH$ *defined in the natural domain* $D(T) = \{\xi \in \mathcal{D} : QH\xi \in \mathcal{D}\}$*.*

Further references

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