Beyond fractality

Steffen Roch
Technische Universität Darmstadt, Germany

Approximation sequences and stability

Let $A \in L(l^2)$ and

$$P_n : l^2 \to l^2, (x_n)_{n\geq 0} \mapsto (x_0, \ldots, x_{n-1}, 0, 0, \ldots).$$

To solve an operator equation $Au = f$ numerically by the finite sections discretization (FSD), consider the sequence of the equations

$$(P_n A|_{\operatorname{im} P_n}) u_n = P_n f, \quad n = 1, 2, \ldots$$

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$$(P_n A|_{\operatorname{im} P_n}) u_n = P_n f, \quad n = 1, 2, \ldots$$

The sequence $(P_n A|_{\operatorname{im} P_n})$ is stable if there is an n_0 such that the $P_n A|_{\operatorname{im} P_n}$ are invertible for $n \geq n_0$ and their inverses are uniformly bounded.

Algebras of approximation sequences

Let $\mathcal{F}$ stand for the set of all bounded sequences (A_n) of operators $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$. Provided with the operations

$$(A_n) + (B_n) = (A_n + B_n),\ (A_n)(B_n) = (A_n B_n),\ (A_n)^* = (A_n^*)$$

and the supremum norm, $\mathcal{F}$ becomes a C^*-algebra, and $\mathcal{G} = \{(A_n) : \|A_n\| \to 0\}$ is a closed ideal of $\mathcal{F}$.

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Stability theorem (Kozak)

A sequence $(A_n) \in \mathcal{F}$ is stable if and only if the coset $(A_n) + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{G}$.

Example: The algebra of the FSD for Toeplitz operators

- For $a \in C(\mathbb{T})$, with kth Fourier coefficient a_k, the Toeplitz operator $T(a) \in L(l^2)$ is given by its matrix representation $(a_{i-j})_{i,j \geq 0}$.
- The Toeplitz algebra $\mathsf{T}(C)$ is the smallest closed subalgebra of $L(l^2)$ which contains all Toeplitz operators $T(a)$ with $a \in C(\mathbb{T})$.
- The algebra of the FSD for Toeplitz operators $\mathcal{S}(\mathsf{T}(C))$ is the smallest closed subalgebra of $\mathcal{F}$, the algebra of all bounded sequences (A_n) with $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$, which contains all sequences $(P_n T(a) P_n)$ with $a \in C(\mathbb{T})$.

The algebra $\mathcal{S}(\mathsf{T}(C))$ of the FSD for Toeplitz operators

Theorem (Böttcher, Silbermann 1983)

(a) $\mathcal{S}(\mathsf{T}(C))$ consists exactly of all sequences (A_n) where

$$A_n = P_n T(a) P_n + P_n K P_n + R_n L R_n + G_n$$

with $a \in C(\mathbb{T})$, K, L compact and $(G_n) \in \mathcal{G}$. This representation is unique.

(b) A sequence $\mathbf{A} = (A_n) \in \mathcal{S}(\mathsf{T}(C))$ is stable (i.e., $\mathbf{A}/\mathcal{G}$ is invertible) if and only if $W(\mathbf{A}) := \text{s-lim}\, A_n P_n$ and $\widetilde{W}(\mathbf{A}) := \text{s-lim}\, R_n A_n R_n$ are invertible.

Here, $R_n : l^2 \to l^2, (x_n)_{n \geq 0} \mapsto (x_{n-1}, \ldots, x_0, 0, 0, \ldots)$.

Fractal algebras

Given $\eta : \mathbb{N} \to \mathbb{N}$ strictly increasing, let

- $\mathcal{F}_\eta := \{(A_{\eta(n)}) : (A_n) \in \mathcal{F}\}$ the restricted algebra,
- $R_\eta : \mathcal{F} \to \mathcal{F}_\eta$, $(A_n) \mapsto (A_{\eta(n)})$ the restriction mapping.

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Definition of a fractal algebra

Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{F}$.

(a) a homomorphism $W : \mathcal{A} \to \mathcal{B}$ is fractal if

$$\forall \eta \; \exists W_\eta : \mathcal{A}_\eta \to \mathcal{B} \text{ such that } W = W_\eta R_\eta|_{\mathcal{A}}.$$

(b) the algebra $\mathcal{A}$ is fractal if the canonical homomorphism $\pi : \mathcal{A} \to \mathcal{A}/(\mathcal{A} \cap \mathcal{G})$ is fractal.

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Example: $\mathcal{S}(\mathsf{T}(C))$ is fractal.

Properties of sequences in fractal algebras

Let (A_n) be a sequence in a fractal subalgebra of $\mathcal{F}$. Then

- $\lim \|A_n\|$ exists and is equal to $\|(A_n) + \mathcal{G}\|$,
- if $(A_n) = (A_n)^*$, then $\lim \operatorname{spec}(A_n) = \operatorname{spec}((A_n) + \mathcal{G})$,
- $\lim \operatorname{spec}_\varepsilon(A_n) = \operatorname{spec}_\varepsilon((A_n) + \mathcal{G})$,
- $(\mathcal{A} \cap \mathcal{K})/\mathcal{G}$ is a dual algebra,
- and more...

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- and more...

Theorem

A C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ is fractal if and only if the limit $\lim \|A_n\|$ exists for every sequence $(A_n) \in \mathcal{A}$.

The fractal restriction theorem

For every separable C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$, there is a strictly increasing η such that $\mathcal{A}_\eta$ is fractal.

(Proof: Diagonal argument.)

The fractal exhaustion theorem

For every separable C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$, there exists a (finite or infinite) number of strictly increasing sequences $\eta_1, \eta_2, \ldots$ with

$$\eta_i(\mathbb{N}) \cap \eta_j(\mathbb{N}) = \emptyset \quad \text{for } i \neq j \quad \text{and} \quad \cup_i \, \eta_i(\mathbb{N}) = \mathbb{N}$$

such that every restriction $\mathcal{A}_{\eta_i}$ is fractal.

(Proof: Repeated use of the fractal restriction theorem.)

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(Proof: Repeated use of the fractal restriction theorem.)

We call $\mathcal{A}$ piecewise fractal if the number of restrictions in the fractal exhaustion theorem is finite.
$\mathcal{A}$ is quasifractal if every restriction of $\mathcal{A}$ has a fractal restriction.

Example: Full FSD for Block Toeplitz operators

Consider Toeplitz operators $T(a)$ with $a : \mathbb{T} \to \mathbb{C}^{N\times N}$ continuous. Let $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ denote the related algebra of the (full) FSD.

Theorem

(a) $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ consists exactly of all sequences (A_n) where

$$A_n = P_n T(a) P_n + P_n K P_n + R_n L_{\kappa(n)} R_n + G_n$$

with $a \in C(\mathbb{T})^{N\times N}$, K, L_i compact, $(G_n) \in \mathcal{G}$, and $\kappa(n)$ is the remainder of n mod N.

(b) A sequence $\mathbf{A} = (A_n) \in \mathcal{S}(\mathsf{T}(C))$ is stable if and only if $W(\mathbf{A}) :=$ s-lim $A_n P_n$ and $\widetilde{W}_i(\mathbf{A}) :=$ s-lim $R_{nN+i} A_{nN+i} R_{nN+i}$ are invertible.

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Consequently, $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ is piecewise fractal.

A C^*-algebra is called

- elementary if it is isomorphic to $K(H)$ for a Hilbert space H;
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Consequences:

- Lifting theorem,
- splitting of singular values,
- formula for α-numbers....

Example: Continuous functions of Toeplitz operators

Let $X = [0, 1]$ and (ξ_n) a dense sequence in X. Let $\mathcal{S}(X, \mathsf{T}(C))$ stand for the smallest closed C^*-subalgebra of $\mathcal{F}$ which contains all sequences $(P_n A(\xi_n) P_n)$ where $A : X \to \mathsf{T}(C)$ is a continuous function. Clearly, $\mathcal{S}(\mathsf{T}(C)) \subseteq \mathcal{S}(X, \mathsf{T}(C))$.

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Theorem

The algebra $\mathcal{S}(X, \mathsf{T}(C))$ is quasifractal.

(Proof: Every subsequence of (ξ_n) has a convergent subsequence $(\xi_{\eta(n)})$. The restriction $\mathcal{S}(X, \mathsf{T}(C))_\eta$ is fractal.)

The fractal variety of an algebra

- Notation: identify strictly increasing sequences η with their range $\mathbb{M} = \eta(\mathbb{N})$.
- For a C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$, let $\operatorname{fr} \mathcal{A}$ stand for the set of all infinite subsets $\mathbb{M}$ of $\mathbb{N}$ such that the restriction $\mathcal{A}|_{\mathbb{M}}$ is fractal.
- Call $\mathbb{M}_1, \mathbb{M}_2 \in \operatorname{fr} \mathcal{A}$ equivalent if $\mathbb{M}_1 \cup \mathbb{M}_2 \in \operatorname{fr} \mathcal{A}$. Then write $\mathbb{M}_1 \sim \mathbb{M}_2$.

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Goals:

- Describe the fractal variety $(\operatorname{fr} \mathcal{A})^\sim := \operatorname{fr} \mathcal{A}/ \sim$ of $\mathcal{A}$.
- Describe the structure of quasifractal algebras.

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Given a quasifractal algebra $\mathcal{A}$, let $\mathcal{L}(\mathcal{A})$ be the smallest closed complex subalgebra of $l^\infty(\mathbb{N})$ which contains all sequences $(\|A_n\|)$ with $(A_n) \in \mathcal{A}$.

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$\mathcal{L}(\mathcal{A})$ is a commutative C^*-algebra; it is quasiconvergent in the following sense.

Quasiconvergent algebras (I)

A C^*-subalgebra $\mathcal{L}$ of l^∞ is quasiconvergent if for every infinite subset $\mathbb{M}'$ of $\mathbb{N}$ there is an infinite subset $\mathbb{M}$ of $\mathbb{M}'$ such that every sequence in $\mathcal{L}|_{\mathbb{M}}$ converges.
Let $\mathrm{cr}\,\mathcal{L}$ denote the set of all infinite subsets $\mathbb{M}$ of $\mathbb{N}$ such that all sequences in $\mathcal{L}|_{\mathbb{M}}$ converge.

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An algebra $\mathcal{A} \subseteq \mathcal{F}$ is quasifractal if and only if $\mathcal{L}(\mathcal{A}) \subseteq l^\infty$ is quasiconvergent. In this case,

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(Proof: The norm limit criterium for fractality.)

Quasiconvergent algebras (II)

For every $\mathbb{M} \in \operatorname{cr} \mathcal{L}$, the mapping

$$\varphi_{\mathbb{M}} : \mathcal{L} \to \mathbb{C}, \quad a \mapsto \lim(a|_{\mathbb{M}})$$

is a multiplicative linear functional on $\mathcal{L}$ which is a character if $\mathbb{M}$ is non-degenerated, i.e. $\mathcal{L}|_{\mathbb{M}} \not\subseteq c_0|_{\mathbb{M}}$. Since $\mathcal{L} \cap c_0$ is in the kernel of this functional, the quotient mapping

$$\varphi_{\mathbb{M}} : \mathcal{L}/(\mathcal{L} \cap c_0) \to \mathbb{C}, \quad a + (\mathcal{L} \cap c_0) \mapsto \lim(a|_{\mathbb{M}}) \qquad (1)$$

is well defined. This mapping is a character of $\mathcal{L}/(\mathcal{L} \cap c_0)$ if $\mathbb{M}$ is non-degenerated.

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Theorem

Let $\mathcal{L}$ be a unital, separable and quasiconvergent C^*-subalgebra of l^∞. Then $\{\varphi_{\mathbb{M}} : \mathbb{M} \in \operatorname{cr} \mathcal{L}\} = \operatorname{Max}(\mathcal{L}/(\mathcal{L} \cap c_0))$.

Sketch of the Proof.

Step 1: $\{\varphi_{\mathbb{M}} : \mathbb{M} \in \operatorname{cr}\mathcal{L}\}$ is strictly spectral.

If $a + (\mathcal{L} \cap c_0)$ is not invertible in $\mathcal{L}/(\mathcal{L} \cap c_0)$, then $a + c_0$ is not invertible in $\mathcal{L}/c_0$. Let $\mathbb{M}'$ be an infinite subset of $\mathbb{N}$ such that $a|_{\mathbb{M}'} \to 0$. Since $\mathcal{L}$ is quasiconvergent, there is an infinite subset $\mathbb{M}$ of $\mathbb{M}'$ which belongs to $\operatorname{cr}\mathcal{L}$. The character associated with $\mathbb{M}$ satisfies $\varphi_{\mathbb{M}}(a) = 0$. Conversely, if $a \in \mathcal{L}$ and $\varphi_{\mathbb{M}}(a) \neq 0$ for all $\mathbb{M} \in \operatorname{cr}\mathcal{L}$, then $a + (\mathcal{L} \cap c_0)$ is invertible in $\mathcal{L}/(\mathcal{L} \cap c_0)$.

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Step 2: $\{\varphi_{\mathbb{M}} : \mathbb{M} \in \mathrm{cr}\,\mathcal{L}\}$ is exhausting.

By a theorem of Nistor/Prudhon (Preprint 2014), every strictly spectral family on a separable C^*-algebra is exhausting. (In the concrete setting, there is a simple direct proof.)

Quasiconvergent algebras (III)

Let $\mathcal{L} \subseteq l^\infty$. Which sets $\mathbb{M} \in \operatorname{cr} \mathcal{L}$ generate the same character?

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Call $\mathbb{M}_1, \mathbb{M}_2 \in \operatorname{cr} \mathcal{L}$ equivalent if $\mathbb{M}_1 \cup \mathbb{M}_2 \in \operatorname{cr} \mathcal{L}$. Then write $\mathbb{M}_1 \sim \mathbb{M}_2$.

By this definition, the mapping

$$(\operatorname{cr} \mathcal{L})^\sim \to \{\varphi_\mathbb{M} : \mathbb{M} \in \operatorname{cr} \mathcal{L}\}, \quad \mathbb{M}^\sim \mapsto \varphi_\mathbb{M}$$

is a (well defined) bijection. Combining this observation with the previous theorem we obtain:

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Theorem

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The fractal variety as a compact Hausdorff space...

Let $\mathcal{A}$ be a unital and quasifractal C^*-subalgebra of $\mathcal{F}$ such that $\mathcal{L}(\mathcal{A})$ is separable. Then

- $\mathrm{fr}\,\mathcal{A} = \mathrm{cr}\,\mathcal{L}(\mathcal{A})$,
- $(\mathrm{fr}\,\mathcal{A})^\sim = (\mathrm{cr}\,\mathcal{L}(\mathcal{A}))^\sim$,
- $\mathcal{L}(\mathcal{A})$ is separable and quasiconvergent.

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- $(\mathrm{fr}\,\mathcal{A})^\sim = (\mathrm{cr}\,\mathcal{L}(\mathcal{A}))^\sim$,
- $\mathcal{L}(\mathcal{A})$ is separable and quasiconvergent.

Thus, there is a (well defined) bijection

$$(\mathrm{fr}\,\mathcal{A})^\sim \to \mathrm{Max}\,(\mathcal{L}(\mathcal{A})/(\mathcal{L}(\mathcal{A}) \cap c_0)), \quad \mathbb{M}^\sim \mapsto \varphi_{\mathbb{M}}$$

which allows to transfer the Gelfand topology of $\mathrm{Max}\,(\mathcal{L}(\mathcal{A})/(\mathcal{L}(\mathcal{A}) \cap c_0))$ onto $(\mathrm{fr}\,\mathcal{A})^\sim$, making the latter to a compact Hausdorff space.

... and quasifractal algebras as continuous fields (I)

Let X be a compact Hausdorff space and $\mathcal{B}$ be the direct product of a family $\{\mathcal{B}_x\}_{x \in X}$ of C^*-algebras. A continuous field of C^*-algebras over X is a C^*-subalgebra $\mathcal{C}$ of $\mathcal{B}$ such that:

(a) $\mathcal{C}$ is maximal, i.e., $\mathcal{B}_x = \{c(x) : c \in \mathcal{C}\}$ for every $x \in X$,
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Let $\mathcal{A}$ be a unital and quasifractal C^*-subalgebra of $\mathcal{F}$ for which $\mathcal{L}(\mathcal{A})$ is separable. Set $X = (\operatorname{fr}\mathcal{A})^{\sim}$, for $\mathbb{M} \in \operatorname{fr}\mathcal{A}$ define $\mathcal{B}_{\mathbb{M}}$ as $\mathcal{A}|_{\mathbb{M}}/(\mathcal{A}|_{\mathbb{M}} \cap \mathcal{G}|_{\mathbb{M}})$, and let $\mathcal{B}$ be the direct product of the family $\{\mathcal{B}_{\mathbb{M}}\}_{\mathbb{M}\in\operatorname{fr}\mathcal{A}}$. Every sequence $\mathbf{A} \in \mathcal{A}$ determines a function in $\mathcal{B}$ via

$$\mathbb{M} \mapsto \mathbf{A}|_{\mathbb{M}} + (\mathcal{A}|_{\mathbb{M}} \cap \mathcal{G}|_{\mathbb{M}}) \in \mathcal{A}|_{\mathbb{M}}/(\mathcal{A}|_{\mathbb{M}} \cap \mathcal{G}|_{\mathbb{M}}). \qquad (2)$$

Let $\mathcal{C}$ be the set of all functions (2) with $\mathbf{A} \in \mathcal{A}$.

... and quasifractal algebras as continuous fields (II)

Theorem

Let $\mathcal{A}$ be a unital and quasifractal C^*-subalgebra of $\mathcal{F}$ for which $\mathcal{L}(\mathcal{A})$ is separable. Then

(a) $\mathcal{C}$ is a continuous field of C^*-algebras over $(\mathrm{fr}\,\mathcal{A})^\sim$,
(b) the mapping which sends $\mathbf{A} + (\mathcal{A} \cap \mathcal{G})$ to the function (2) is a *-isomorphism from $\mathcal{A}/(\mathcal{A} \cap \mathcal{G})$ onto $\mathcal{C}$.

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Thank you for your attention.