Mapping theorems for Sobolev spaces of vector-valued functions (joint work with Wolfgang Arendt)

Marcel Kreuter | August 14, 2017 | Ulm University – Institute of Applied Analysis

Motivation

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Lemma

$u \in H^1(0,t;L^2(\Omega)) \;\Rightarrow\; u(\cdot)^+ \in H^1(0,t;L^2(\Omega))$
$D_j u(s)^+ = D_j u(s) \cdot 1_{\{u(s)>0\}}$

W. Arendt, D. Dier, M. Kramar Fijavž – Diffusion in networks with time-dependent transmission conditions, Appl. Math. Optim., 69:315–336 (2014)

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Lemma

$u \in H^1(\Omega;H) \;\Rightarrow\; P_C u(\cdot) \in H^1(\Omega;H)$

S. Cardanobile, D. Mugnolo – Parabolic systems with coupled boundary conditions, J. Differ. Equ. 247:1229–1248 (2009)

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Let X and Y be Banach spaces, $\Omega \subset \mathbb{R}^d$ open and $1 \leq p \leq \infty$.

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Suppose $F : X \to Y$ is Lipschitz continuous. Under which circumstances does F induce an Operator (Nemytskii- or superposition operator)

$$\begin{aligned} W^{1,p}(\Omega, X) &\to W^{1,p}(\Omega, Y) \\ u &\mapsto F \circ u \qquad ? \end{aligned} \tag{$*$}$$

A counter example

$$\begin{aligned} u : (0,1) &\to C([0,1]) \\ u(t)(r) &= r - t \end{aligned}$$

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$$\frac{d}{dt} u^+(t)(r) \;\text{"="}\; \begin{cases} 0, & \text{if } r < t \\ -1, & \text{if } r \geq t \end{cases}$$

Theorem (Arendt, K., 2017)

$(*)$ *holds for all Lipschitz continuous* $F : X \to Y$ *(*$|\Omega| < \infty$ *or* $F(0) = 0$*)*

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Note: Y *has the Radon-Nikodym property iff every Lipschitz continuous (or equivalently every absolutely continuous) function* $f : \mathbb{R} \to Y$ *is differentiable a.e.*

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+ interesting corollaries

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Proof.

$u \in W^{1,p}(\Omega,X)$

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$\Rightarrow \|u(\cdot)\|_X \in W^{1,p}(\Omega, \mathbb{R})$

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$\Rightarrow u \in L^{p^*}(\Omega, X)$ □

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- $(*)$ almost never needed for all F

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- $(*)$ almost never needed for all $F \Rightarrow$ too broad for applications

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Let $F : X \to Y$ be Lipschitz continuous AND one-sided Gateaux differentiable ($|\Omega| < \infty$ or $F(0) = 0$), then $(*)$ holds.

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Let $F : X \to Y$ be Lipschitz continuous AND one-sided Gateaux differentiable ($|\Omega| < \infty$ or $F(0) = 0$), then $(*)$ holds.

Further $D_j(F \circ u(\xi)) = D^+_{D_j u(\xi)} F(u(\xi)) = D^-_{D_j u(\xi)} F(u(\xi))$ a.e.

Examples

Seen before: $\|u(\cdot)\|_X \in W^{1,p}(\Omega, \mathbb{R})$ for all $u \in W^{1,p}(\Omega, X)$

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where $J(x) := \{x' \in X', \|x'\| = 1, \langle x, x' \rangle = \|x\|_X\}$

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$$\Rightarrow D_j \|u(\xi)\|_X = \langle D_j u(\xi), x' \rangle$$

for all $x' \in J(u(\xi))$

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$$D_j u(\cdot)^+ = P_{u^+(\cdot)} D_j u(\cdot)$$

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$$A : D(A) \subset L^p(\Omega, \mathbb{R}) \to L^p(\Omega, \mathbb{R}) \text{ sectorial}$$
$$\exists! \tilde{A} : D(\tilde{A}) \subset L^p(\Omega, H) \to L^p(\Omega, H)$$
$$R(\lambda, \tilde{A})(f \otimes x) = R(\lambda, A)f \otimes x \ \forall \lambda < 0$$

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Further B has compact resolvent and $D(A) \subset W^{1,p}(\Omega, \mathbb{R})$.

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Suppose that A and B have bounded imaginary powers,

$$\varphi_{\mathrm{bip}} A + \varphi_{\mathrm{bip}} B < \pi$$

Further B has compact resolvent and $D(A) \subset W^{1,p}(\Omega, \mathbb{R})$. Then

$$\tilde{A} + \tilde{B} : D(\tilde{A}) \cap D(\tilde{B}) \to L^p(\Omega, H)$$

is sectorial and has compact resolvent.

Reference

W. Arendt, M. K. – Mapping theorems for Sobolev spaces of vector-valued functions, to appear in Studia Math. (2017) arXiv:1611.06161