On Birman's Sequence of Hardy-Rellich Type Inequalities

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Introduction

In 1961, **M. Š. Birman** established the following sequence of integral inequalities:

M. Š Birman (1961)

For $n \in \mathbb{N}$ and $f \in C_0^n((0,\infty))$,

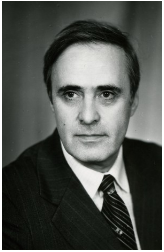
$$\int_0^\infty \left|f^{(n)}(x)\right|^2 dx \geq \frac{((2n-1)!!)^2}{2^{2n}} \int_0^\infty \left|\frac{f(x)}{x^n}\right|^2 dx. \qquad (I_n)$$

In particular, I_1 is the classical **Hardy inequality**

$$\int_0^\infty \left|f'(x)\right|^2 dx \geq \frac{1}{4} \int_0^\infty \left|\frac{f(x)}{x}\right|^2 dx,$$

and I_2 is the **Rellich inequality**

$$\int_0^\infty \left|f''(x)\right|^2 dx \geq \frac{9}{16} \int_0^\infty \left|\frac{f(x)}{x^2}\right|^2 dx.$$



Introduction

Our joint paper shows:

- A **new proof** of Birman's inequalities on a more general Hilbert space $H_n([0,\infty))$ of functions on $[0,\infty)$.
- For any $0 < b < \infty$, these inequalities hold on the **standard Sobolev space** $H_0^n((0,b))$.
- Birman's constants $((2n-1)!!)^2/2^{2n}$ in these inequalities are **best possible** and the only function that gives equality is the function identically zero in $L^2((0,\infty))$.
- Birman's inequalities are closely related to a sequence of generalized **continuous Cesáro operators**, $\{T_n\}$, with interesting spectral properties.
- This generalized sequence of inequalities extends mutatis mutandis to $\mathcal{H}$-valued functions, where $\mathcal{H}$ is a **separable Hilbert space**.

The Function Spaces $H_n([0,\infty))$ and $H_n((0,\infty))'$

Definition 1 (The Function Space $H_n([0,\infty))$)

Let $n \in \mathbb{N}$. Define the function space $H_n([0,\infty))$ via

$$H_n([0,\infty)) := \big\{ f : [0,\infty) \to \mathbb{C} \,\big|\, f^{(j)} \in AC_{\mathrm{loc}}([0,\infty)); f^{(n)} \in L^2((0,\infty)); \\ f^{(j)}(0) = 0,\ j = 0, 1, \ldots, n-1 \big\}.$$

Note: **G. H. Hardy**, **J. E. Littlewood**, and **G. Pólya** proved the classical Hardy inequality I_1 on H_1 in 1934.

The fact

$$f \in H_n([0,\infty)) \Rightarrow f' \in H_{n-1}([0,\infty)).$$

is important in the new proof of Birman's inequalities.

When endowed with the inner product

$$(f,g)_{H_n([0,\infty))} := \int_0^\infty \overline{f^{(n)}}(x) g^{(n)}(x) dx,$$

$H_n([0,\infty))$ is a **Hilbert space**.

The Function Spaces $H_n([0,\infty))$ and $H_n((0,\infty))'$

Proposition 1

The inner product space $(H_n([0,\infty)), (\cdot,\cdot)_{H_n([0,\infty))})$ is actually a Hilbert space. In addition, $C_0^\infty((0,\infty))$ is dense in $(H_n([0,\infty)), (\,\cdot\,,\,\cdot\,)_{H_n([0,\infty))})$.

Caution: We emphasize that

$$H_n([0,\infty)) \neq H_0^n((0,\infty)), \quad n \in \mathbb{N},$$

with $H_0^n((0,\infty))$ the **standard Sobolev space** obtained upon completing $C_0^\infty((0,\infty))$ in the norm of $H^n((0,\infty))$.

Indeed, define $\widetilde{f} \in H_n([0,\infty))$ via,

$$\widetilde{f}(x) = \begin{cases} 0, & x \text{ near } 0, \\ x^{(2n-1)/2}/\ln(x), & x \text{ near } \infty, \end{cases}$$

such that

$$\widetilde{f}^{(j)} \in AC_{loc}([0,\infty)), \quad j = 0, 1, \dots, n.$$

Calculations show $\widetilde{f}^{(j)} \notin L^2((0,\infty))$, $0 \leq j \leq n-1$.

The Function Spaces $H_n([0,\infty))$ and $H_n((0,\infty))'$

Theorem 2 is used in the new proof of Birman's inequalities.

Theorem 2

Let $f \in H_n([0,\infty))$. *Then*

(i) $f^{(n-j)}/x^j \in L^2((0,\infty))$ *for* $j = 0, 1, \ldots n$; *In particular,* $f \in H_n([0,\infty)) \implies f' \in H_{n-1}([0,\infty))$;

(ii) $\lim_{x\to\infty} \dfrac{\left(f^{(j)}(x)\right)^2}{x^{2n-2j-1}} = 0, \quad j = 0, 1, \ldots, n-1$;

(iii) $\lim_{x\downarrow 0} \dfrac{\left(f^{(j)}(x)\right)^2}{x^{2n-2j-1}} = 0, \quad j = 0, 1, \ldots, n-1.$

The above is proved using an integral inequality independently due to **G. Tomaselli** (1969), **G. Talenti** (1969), **R. S. Chisholm** & **W. N. Everitt** (1971), and **B. Muckenhoupt** (1972).

The Function Spaces $H_n([0,\infty))$ and $H_n((0,\infty))'$

Consider the spaces $H_n((0,\infty))'$ and $D_n([0,\infty))$ given below.

Definition 3 (The Function Space $H_n((0,\infty))'$)

Let $n \in \mathbb{N}$. Define the function space $H_n((0,\infty))'$ via

$$H_n((0,\infty))' := \left\{ f : (0,\infty) \to \mathbb{C} \,\middle|\, f^{(j)} \in AC_{\mathrm{loc}}((0,\infty)), \right.$$
$$\left. j = 0, 1, \ldots, n-1;\ f^{(n)}, f/x^n \in L^2((0,\infty)) \right\}.$$

Definition 4 (The Function Space $D_n([0,\infty))$)

Let $n \in \mathbb{N}$. Define the function space $D_n([0,\infty))$ via

$$D_n([0,\infty)) := \left\{ \int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t)\,dt\,dt_{n-1}\ldots dt_1 \,\middle|\, f \in L^2((0,\infty)) \right\}$$

Surprisingly, $H_n([0,\infty))$ is equal to both spaces.

Theorem 5

For each $n \in \mathbb{N}$, $H_n([0,\infty)) = H_n((0,\infty))' = D_n([0,\infty))$.

A New Proof of Birman's Hardy-Rellich Type Inequalities

Theorem 6 (Birman's Inequalities on $H_n([0,\infty))$)

Let $n \in \mathbb{N}$ and $0 \neq f \in H_n([0,\infty))$. Then,

$$\int_0^\infty \left| f^{(n)}(x) \right|^2 dx > \frac{((2n-1)!!)^2}{2^{2n}} \int_0^\infty \left| \frac{f(x)}{x^n} \right|^2 dx.$$

Our new proof of Birman's inequalities consists of iterating Hardy's inequality, with repeated use of the elementary inequality

$$2xy \leq \varepsilon x^2 + \varepsilon^{-1} y^2, \quad x, y \in \mathbb{R}, \varepsilon > 0.$$

This, integration by parts, and the Cauchy–Schwarz inequality, results in

$$\int_0^\infty \left| f^{(n)}(x) \right|^2 dx \geq \begin{cases} \left(-\varepsilon^2 + \varepsilon\right) \int_0^\infty \frac{|f(x)|^2}{x^2} dx, & n = 1, \\ \frac{(2n-3)!!}{2^{2n-2}} \left(-\varepsilon^2 + (2n-1)\varepsilon\right) \int_0^\infty \frac{|f(x)|^2}{x^{2n}} dx, & n \geq 2. \end{cases}$$

Maximizing over $\varepsilon \in (0,\infty)$ proves the theorem.

Optimality of Birman's Constant

Theorem 7 (Optimality of Birman's Constant)

The constant $((2n-1)!!)^2/2^{2n}$ *in Birman's inequalities is* **best possible** *on* $H_n([0,\infty))$ *for all* $n \in \mathbb{N}$.

Recalling $D_n([0,\infty)) = H_n([0,\infty))$, where

$$D_n([0,\infty)) := \left\{ \int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t)\,dt\,dt_{n-1}\ldots dt_1 \;\middle|\; f \in L^2((0,\infty)) \right\},$$

leads to the construction of an interesting linear operator T_n.

Definition 8 (The Linear Operator T_n)

Let $n \in \mathbb{N}$. Define the linear operator T_n on $L^2((0,\infty))$ via

$$(T_n f)(x) := \frac{1}{x^n} \int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t)\,dt\,dt_{n-1}\ldots dt_1, \quad f \in L^2((0,\infty)).$$

Note: $T_n f \in L^2((0,\infty)), \quad f \in L^2((0,\infty))$ by Thms. 5, 2 (i).

Generalized Continuous Césaro Operators T_n

The operator T_n is a generalization of the **continuous Césaro operator** on $L^2((0,\infty))$,

$$(T_1 f)(x) = \frac{1}{x}\int_0^x f(x)dx, \quad f \in L^2((0,\infty))$$

also known as the classical **Hardy (integral) operator**.

Birman's inequalities are closely related to T_n, which posseses several interesting properties of its own.

Theorem 9 (Boundedness and Non-Compactness of T_n)

Let $n \in \mathbb{N}$ and define T_n as above. Then T_n is bounded in $L^2((0,\infty))$ with operator norm

$$\|T_n\| = \frac{2^n}{(2n-1)!!}.$$

T_n is not compact (it has purely a.c. spectrum).

Generalized Continuous Césaro Operators T_n

Theorem 10 (Invertibility of T_n)

Define T_n, $n \in \mathbb{N}$, as above. Then T_n is invertible and

$$\operatorname{dom}\left(T_n^{-1}\right) = \operatorname{dom}\left(T_1^{-n}\right) = \Big\{ f \in L^2((0,\infty)) \,\Big|\, f \in AC_{loc}^{(n-1)}((0,\infty)); \; x^j f^{(j)} \in L^2((0,\infty)),\, j = 1, \dots, n \Big\},$$

$$\left(T_n^{-1} f\right)(x) = \frac{d^n}{dx^n} x^n f(x), \quad f \in \operatorname{dom}\left(T_n^{-1}\right).$$

Note: T_n is not *boundedly* invertible as $0 \in \sigma(T_n)$, $n \in \mathbb{N}$.

For $g = (T_n \circ T_n^{-1})g$, it is necessary that $\lim_{x \to 0^+} (x^n g(x))^{(j)} = 0$, $0 \le j \le n-1$. Surprisingly, this is *consequence* of lying in the space above.

Lemma 11

Let $n \in \mathbb{N}$. Assume $f \in AC_{loc}^{(n-1)}((0,\infty))$ and $x^k f^{(k)} \in L^2((0,\infty))$ for $k = 0, 1, \dots, n$. Then

$$\lim_{x \downarrow 0} (x^n f(x))^{(j)} = 0, \quad j = 0, 1, \dots, n-1.$$

Generalized Continuous Césaro Operators T_n

We introduce the unitary **Mellin transform**, $\mathcal{M}$, given by

$$\mathcal{M}\colon \begin{cases} L^2((0,\infty);dx) \to L^2(\mathbb{R};d\lambda), \\ f \mapsto (\mathcal{M}f)(\lambda) \equiv f^*(\lambda) := (2\pi)^{-1/2}\,\text{s-lim}_{a\to\infty} \int_{1/a}^{a} f(x) x^{-(1/2)+i\lambda}dx \\ \hfill \text{for a.e. } \lambda\in\mathbb{R}, \end{cases}$$

$$\mathcal{M}^{-1}\colon \begin{cases} L^2(\mathbb{R};d\lambda) \to L^2((0,\infty);dx), \\ f^* \mapsto (\mathcal{M}^{-1}f^*)(x) \equiv f(x) := (2\pi)^{-1/2}\,\text{s-lim}_{b\to\infty} \int_{-b}^{b} f^*(\lambda) x^{-(1/2)-i\lambda}d\lambda \\ \hfill \text{for a.e. } x\in(0,\infty). \end{cases}$$

The fact,

$$i\left(\frac{d}{dx}x - \frac{1}{2}\right)x^{-(1/2)-i\lambda} = \lambda x^{-(1/2)-i\lambda}, \quad x\in(0,\infty),\ \lambda\in\mathbb{R},$$

leads to the following definition of the operator S_1 in $L^2((0,\infty);dx)$,

$$S_1 := i\big(T_1^{-1} - 2^{-1}I_{L^2((0,\infty))}\big), \quad \operatorname{dom}(S_1) = \operatorname{dom}\left(T_1^{-1}\right),$$

and shows S_1 is **unitarily equivalent** to the operator of multiplication by the independent variable in $L^2(\mathbb{R};d\lambda)$,

$$\big(\mathcal{M}S_1\mathcal{M}^{-1}f^*\big)(\lambda) = \lambda f^*(\lambda) \text{ for a.e. } \lambda\in\mathbb{R} \text{ and}$$

$$\text{for all } f^*\in L^2(\mathbb{R};d\lambda) \text{ such that } \lambda f^*\in L^2(\mathbb{R};d\lambda).$$

Generalized Continuous Césaro Operators T_n

Summarizing, the Mellin transform **diagonalizes** S_1 and hence T_1. Thus, the spectrum, and normality, of T_1 can be determined through study of S_1.

Theorem 12

Define S_1 as above. Then S_1 is self-adjoint and hence T_1 is normal.

Moreover, the spectra of S_1 and T_1 are **simple** *and* **purely absolutely continuous**.

In particular,

$$\sigma(S_1) = \sigma_{ac}(S_1) = \mathbb{R}, \quad \sigma(T_1) = \sigma_{ac}(T_1) = C(1;1).$$

Here $C(z_0; r_0) \subset \mathbb{C}$ denotes the circle of radius $r_0 > 0$ centered at $z_0 \in \mathbb{C}$.

Note: The spectrum of T_1 was originally computed by **A. Brown**, **P. R. Halmos**, and **A. L. Shields** in 1965.

These preliminary results are important in determining the spectral properties of T_n for all $n \in \mathbb{N}$.

Generalized Continuous Césaro Operators T_n

For $n \in \mathbb{N}$, let p_n be the polynomial given by

$$p_n(z) = \prod_{k=0}^{n-1}(z+k), \quad z \in \mathbb{C},$$

and r_n the rational function given by

$$r_n(z) = z^n \prod_{k=1}^{n-1}(1+kz)^{-1}, \quad z \in \mathbb{C}\backslash\{-\ell^{-1}\}_{1\leq\ell\leq n-1}.$$

Theorems 10, 12, and the **spectral mapping theorem**, yield

Theorem 13 (Spectrum of T_n)

Let $n \in \mathbb{N}$ and define p_n and r_n as above. Then T_n is normal and

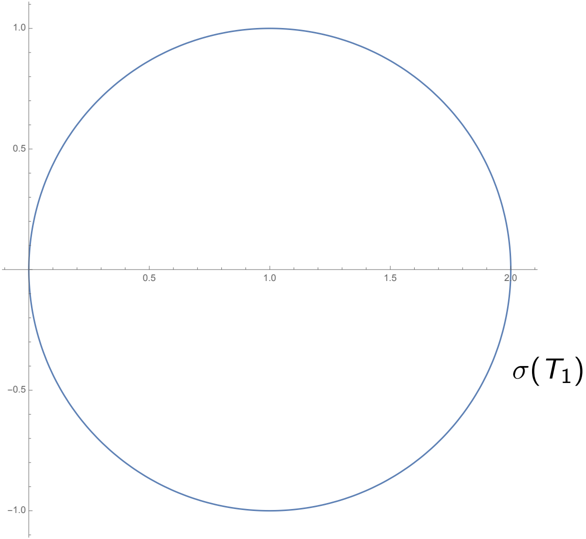
(i) $T_n^{-1} = p_n(T_1^{-1})$, $\quad \sigma(T_n^{-1}) = \sigma_{ac}(T_n^{-1}) = p_n(\sigma(T_1^{-1}))$.

(ii) $T_n = r_n(T_1)$, $\quad \sigma(T_n) = \sigma_{ac}(T_n) = r_n(\sigma(T_1))$.

Moreover,

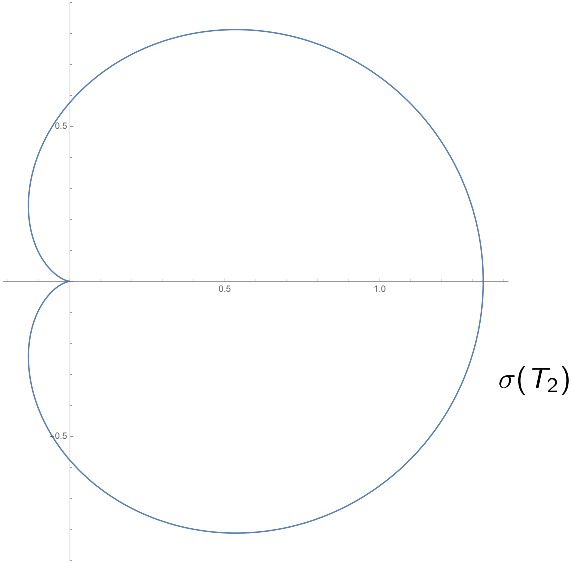
$$\sigma_p(T_n) = \sigma_{sc}(T_n) = \emptyset.$$

Thus, $\sigma(T_n)$ is a direct consequence of $\sigma(T_1)$ for all $n \in \mathbb{N}$.



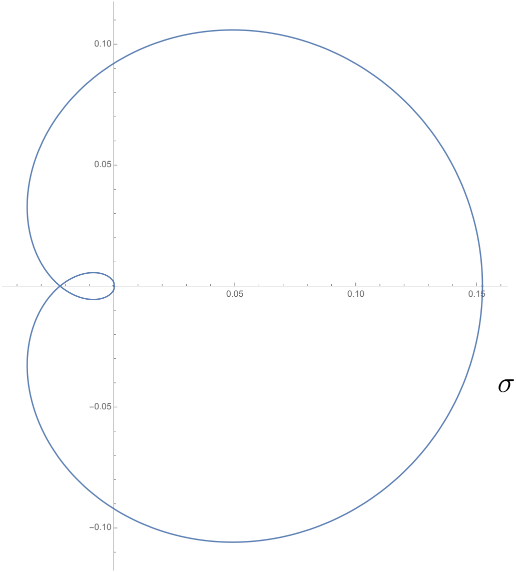
$n = 1$

Generalized Continuous Césaro Operators T_n



$n = 2$

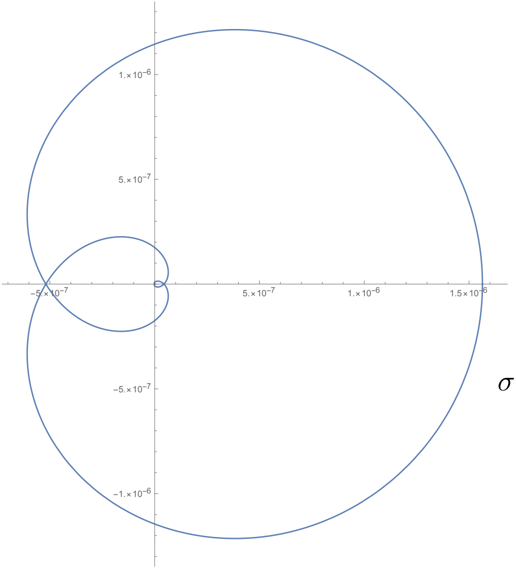
Generalized Continuous Césaro Operators T_n



$\sigma(T_4)$

$n = 4$

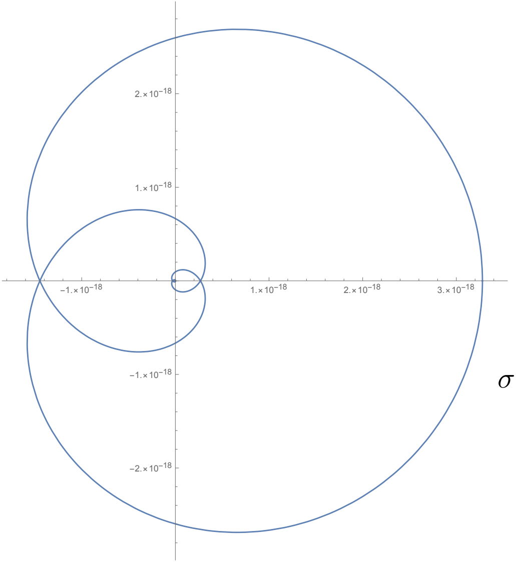
Generalized Continuous Césaro Operators T_n



$\sigma(T_{10})$

$n = 10$

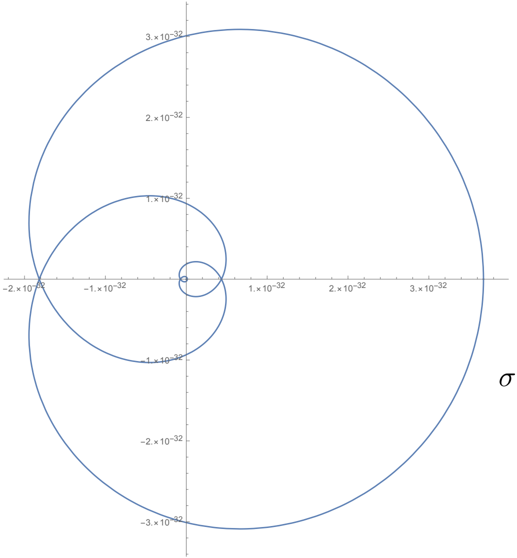
Generalized Continuous Césaro Operators T_n



$\sigma(T_{20})$

$n = 20$

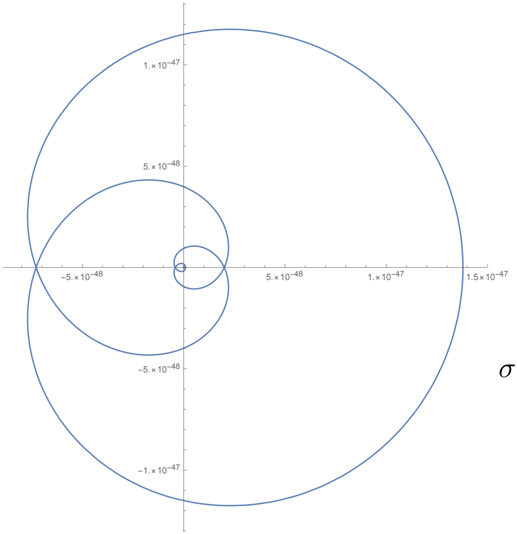
Generalized Continuous Césaro Operators T_n



$\sigma(T_{30})$

$n = 30$

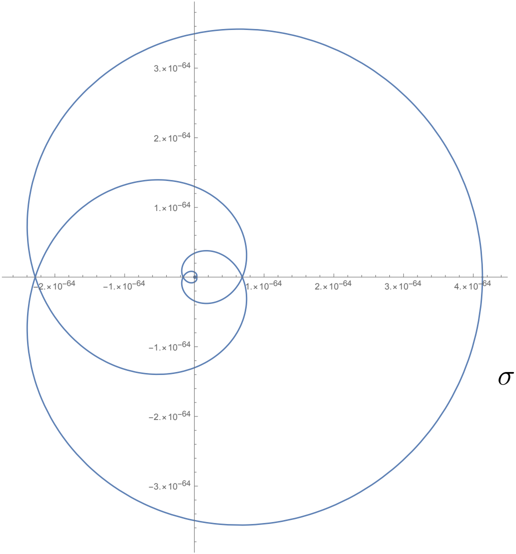
Generalized Continuous Césaro Operators T_n



$\sigma(T_{40})$

$n = 40$

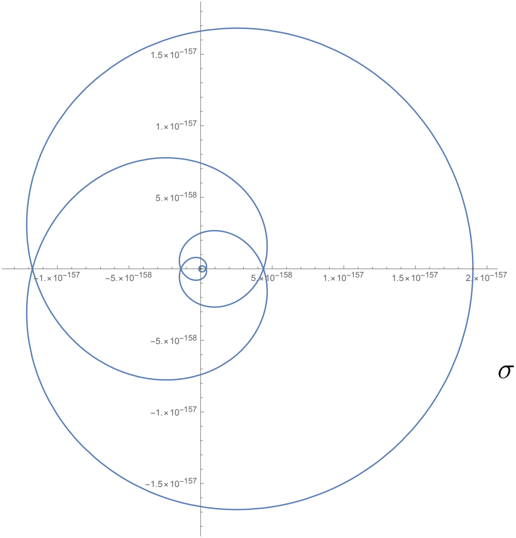
Generalized Continuous Césaro Operators T_n



$\sigma(T_{50})$

$n = 50$

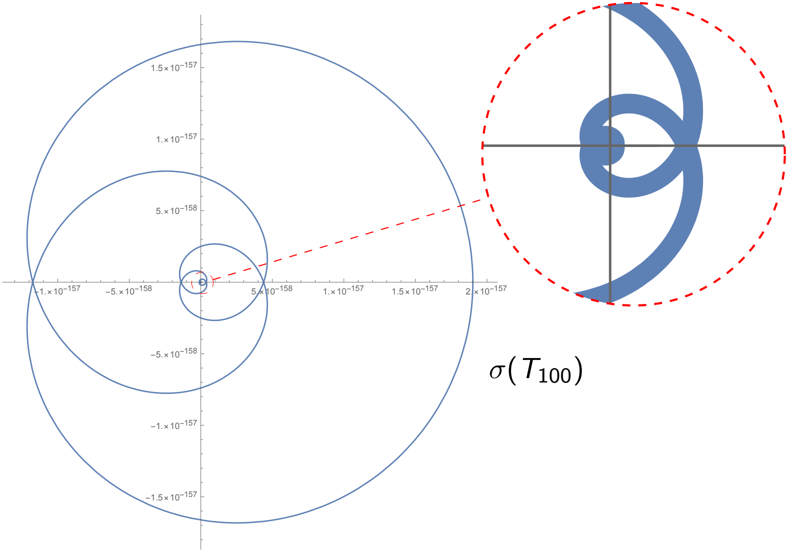
Generalized Continuous Césaro Operators T_n



$\sigma(T_{100})$

$n = 100$

Generalized Continuous Césaro Operators T_n

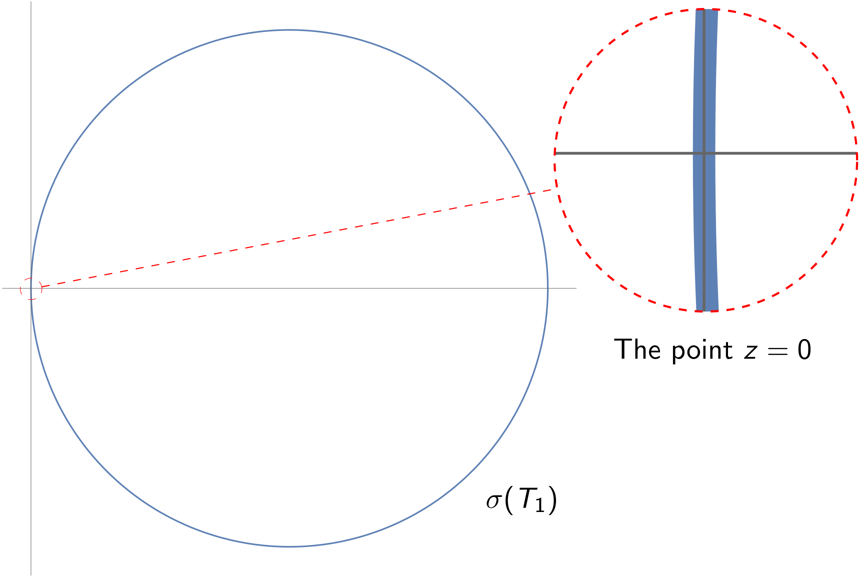


$\sigma(T_{100})$

$n = 100$

Generalized Continuous Césaro Operators T_n

Spectrum of T_n for $n = 1, \ldots, 100$



Birman's Inequalities on the Finite Interval $[0, b]$

We now consider Birman's inequalities for functions defined on the finite interval $[0, b]$ for $0 < b < \infty$.

Definition 14 (The Function Space $H_n([0, b])$)

Let $n \in \mathbb{N}$, $0 < b < \infty$. Define the function space $H_n([0, b])$ via

$$H_n([0,b]) := \big\{ f : [0,b] \to \mathbb{C} \,\big|\, f^{(n)} \in L^2((0,b));\ f^{(j)} \in AC([0,b]); \\ f^{(j)}(0) = f^{(j)}(b) = 0,\ j = 0, 1, \dots, n-1 \big\}.$$

Definition 15 (The Function Space $H_n((0, b])')$

Let $n \in \mathbb{N}$, $0 < b < \infty$. Define the function space $H_n((0, b])'$ via

$$H_n((0,b])' := \big\{ f : (0,b] \to \mathbb{C} \,\big|\, f^{(n)}, f/x^n \in L^2((0,b)); \\ f^{(j)} \in AC_{loc}((0,b]);\ f^{(j)}(b) = 0,\ j = 0, 1, \dots, n-1 \big\}.$$

Birman's Inequalities on the Finite Interval $[0, b]$

Let $H_0^n((0,b))$ be the **standard Sobolev space** on $(0,b)$ obtained upon completion of $C_0^\infty((0,b))$ in the norm of $H^n((0,b))$. That is,

$$H^n((0,b)) = \left\{ f : [0,b] \to \mathbb{C} \,\middle|\, f \in AC^{(n-1)}([0,b]); f^{(k)} \in L^2((0,b)),\ k = 0,1,\ldots,n \right\},$$

and

$$H_0^n((0,b)) = \left\{ f \in H^n((0,b)) \,\middle|\, f^{(j)}(0) = f^{(j)}(b) = 0,\ j = 0,1,\ldots,n-1 \right\}.$$

Using the boundary conditions $f^{(j)}(0) = f^{(j)}(b) = 0$, and the **Friedrichs inequality**,

$$\left\| f^{(j)} \right\|_{L^2((0,b))} \leq C \left\| f^{(n)} \right\|_{L^2((0,b))}, \quad f \in H_0^n((0,b));$$

for $0 \leq j \leq n$, calculations show

$$H_n([0,b]) = H_n((0,b])' = H_0^n((0,b)).$$

and Birman's inequalities hold on $H_n([0,b])$ for any $0 < b < \infty$.

The Vector-Valued Case

All results extend to $\mathcal{H}$-**valued functions**, with $\mathcal{H}$ a **separable, complex Hilbert space**.

Defining all previous spaces analogously for $f : I \to \mathcal{H}$ where $I = [0,\infty), (0,\infty), [0,b], (0,b]$, with $0 < b < \infty$, shows

Theorem 16

For each $n \in \mathbb{N}$, $H_n([0,\infty);\mathcal{H}) = H_n((0,\infty);\mathcal{H})' = D_n([0,\infty);\mathcal{H})$.

Theorem 17 (Birman's Inequalities on $H_n([0,\infty);\mathcal{H})$)

For $0 \neq f \in H_n([0,\infty);\mathcal{H})$,

$$\int_0^\infty \left\|f^{(n)}(x)\right\|_{\mathcal{H}}^2 dx > \frac{[(2n-1)!!]^2}{2^{2n}} \int_0^\infty \frac{\|f(x)\|_{\mathcal{H}}^2}{x^{2n}} dx, \quad n \in \mathbb{N}.$$

The Birman constant $\frac{[(2n-1)!!]^2}{2^{2n}}$ *is best possible on* $H_n([0,\infty);\mathcal{H})$.

The Vector-Valued Case

Theorem 18 (Birman's Inequalities on $H_n([0,b];\mathcal{H})$)

Let $n \in \mathbb{N}$, $b \in (0,\infty)$, *and define* $H_n([0,b];\mathcal{H})$, $H_n((0,b];\mathcal{H})'$ *as above. Then,*

(i) *For each* $n \in \mathbb{N}$,

$$H_n([0,b];\mathcal{H}) = H_n((0,b];\mathcal{H})' = H_0^n((0,b);\mathcal{H})$$

as sets. In particular,

$$f \in H_n([0,b];\mathcal{H}) \implies f^{(j)} \in L^2((0,b);\mathcal{H}), \quad j = 0,1,\dots,n.$$

The norms in $H_n([0,b];\mathcal{H})$ *and* $H_0^n((0,b);\mathcal{H})$ *are equivalent.*

(ii) *For* $f \in H_0^n((0,b))$ *one has*

$$\int_0^b \left\| f^{(n)}(x) \right\|_{\mathcal{H}}^2 dx \geq \frac{[(2n-1)!!]^2}{2^{2n}} \int_0^b \frac{\|f(x)\|_{\mathcal{H}}^2}{x^{2n}}\, dx;$$

(iii) *The constant* $[(2n-1)!!]^2/2^{2n}$ *is optimal.*

(iv) *If* $0 \neq f \in H_0^n((0,b);\mathcal{H})$ *the inequalities in* (ii) *are strict.*