The Ritt property of subordinated operators in the group case

Christian Le Merdy – IWOTA (Chemnitz), August 18th, 2017

Joint work with F. Lancien

Average operators, subordination

Let X be a Banach space.
Let $B(X)$ be the algebra of bounded operators on X.

Let G be an abelian locally compact group.
Let $\pi\colon G \to B(X)$ be a **representation**, that is,

$$\forall\, s,t \in G,\ \pi(ts) = \pi(t)\pi(s) \quad \text{and} \quad \pi(e) = I_X.$$

Assume that π is **strongly continuous**: for any $x \in X$, $t \mapsto \pi(t)x$ is continuous from G into X.
Assume that π is **bounded**: there exists $C \geqslant 1$ such that $\|\pi(t)\| \leqslant C$ for any $t \in G$.

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Assume that π is **bounded**: there exists $C \geqslant 1$ such that $\|\pi(t)\| \leqslant C$ for any $t \in G$.

To any probability measure $\nu \in M(G)$, we associate the **average operator**

$$S(\pi,\nu) = \int_G \pi(t)\, d\nu(t)\,.$$

Average operators, subordination (continued)

Such average operators appear in many places, especially in ergodic theory.

References : Derriennic-Lin, 1989; Jones-Rosenblatt-Tempelman, 1994; Lin-Wittmann, 1994; Jones-Reinhold, 2001.

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<u>Case $G = \mathbb{Z}$</u>

Let $U \in B(X)$ be an invertible operator with $\sup_{k\in\mathbb{Z}} \|U^k\| < \infty$.
Associate $\pi\colon \mathbb{Z} \to B(X)$ by $\pi(k) = U^k$.
A probability measure on $\mathbb{Z}$ is a sequence $\nu = (c_k)_{k\in\mathbb{Z}}$ with

$$\forall\, k \in \mathbb{Z}, \quad c_k \geqslant 0 \qquad \text{and} \qquad \sum_k c_k = 1.$$

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In this case,

$$S(\nu, \pi) = \sum_{k=-\infty}^{\infty} c_k U^k.$$

Then $S(\nu, \pi)$ is subordinated to U in the sense of Dungey (2011).

Average operators, subordination (continued)

Several recent papers on subordination operators induced by probability measures on the semigroup $\mathbb{N}$ or on the semigroup $\mathbb{R}_+$ motivated this work.

References : Dungey, 2011; Gomilko-Tomilov (2015 + ?); Batty-Gomilko-Tomilov (2017).

Recent papers about the group case: Cohen-Cuny-Lin (2014), Cuny (2016).

Questions

An operator $S \in B(X)$ is called **power bounded** if there exists $C \geqslant 1$ such that

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$$S(\pi, \nu_1) S(\pi, \nu_2) = S(\pi, \nu_1 * \nu_2).$$

Consequently, any $S(\pi, \nu)$ is power bounded.

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Question 1. When is $S(\pi, \nu)$ a Ritt operator?

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Question 1. When is $S(\pi, \nu)$ a Ritt operator?

Question 2. When does $S(\pi, \nu)$ admit a bounded H^∞-functional calculus?

Ritt operators

A **Ritt operator** $S \in B(X)$ is a power bounded operator such that

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Let $\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\}$.
This is equivalent to the so-called Ritt condition:

$$\sigma(S) \subset \overline{\mathbb{D}} \qquad \text{and} \qquad \forall\, z \notin \overline{\mathbb{D}}, \quad \|R(z,S)\| \leqslant \frac{K}{|z-1|}\,.$$

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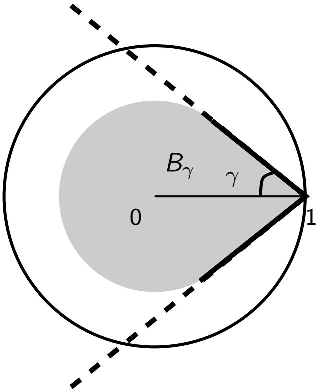
References: Coulhon-Saloff Coste, Nevanlinna, 1993; Lyubich, 1999; Nagy-Zemanek, 1999; Blunck, 2001; etc.

Spectral property of Ritt operators

For any angle $0 < \gamma < \frac{\pi}{2}$, let B_γ be the convex hull of 1 and the disc of center 0 and radius $\sin\gamma$.



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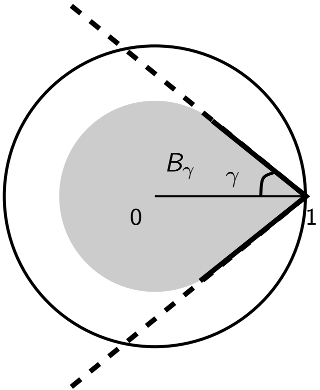

If $S \in B(X)$ is a Ritt operator, then

$$\exists\, 0 < \gamma < \frac{\pi}{2} \quad | \qquad \sigma(S) \subset B_\gamma.$$

H^∞-calculus of Ritt operators

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Let S be a Ritt operator and let $0 < \gamma < \frac{\pi}{2}$.

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We say that S has a **bounded** $H^\infty(B_\gamma)$ **functional calculus** if

$$\sigma(S) \subset B_\gamma$$

and there exists a constant $K \geqslant 1$ such that

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We simply say that S admits a **bounded** H^∞**-functional calculus** if this happens for some $0 < \gamma < \frac{\pi}{2}$.

H^∞-calculus of Ritt operators (continued)

If a Ritt operator $S \in B(X)$ admits a bounded H^∞-functional calculus, then it is polynomially bounded, that is, there exists a constant $K \geqslant 1$ such that

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The converse is **not** true (F. Lancien-Le Merdy, 2015).

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In many contexts, a bounded H^∞-functional calculus for a Ritt operator is equivalent to certain **square function estimates**.

References: Haak-Haase; Le Merdy.

Necessary condition on ν

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Definition. We say that a ν has **BAR** (for 'bounded angular ratio') if there exists $\delta > 1$ such that

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Let $\lambda\colon G \to B(L^2(G))$ be defined by $\lambda(t)f = f(\cdot - t)$ for any $f \in L^2(G)$ and any $t \in G$. Then

$$M_\nu := \int_G \lambda(t)\, d\nu(t) \; = \nu * \cdot : L^2(G) \to L^2(G),$$

this is normal operator and $\sigma(M_\nu) = \overline{\widehat{\nu}(\widehat{G})}$.

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Hence M_ν is Ritt if and only if ν has BAR.

In this case, it automatically admits a bounded H^∞-functional calculus.

Refined questions

Note that

$$M_\nu = S(\lambda, \nu)$$

for the bounded strongly continuous representation $\lambda\colon G \to B(L^2(G))$.

Hence BAR is a natural necessary condition for Questions 1 & 2.

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Hence BAR is a natural necessary condition for Questions 1 & 2.

- Assume that ν has BAR.

(i) Is $S(\pi, \nu) = \int_G \pi(t)\, d\nu(t)$ a Ritt operator for any strongly continuous bounded representation π?

(ii) Which additional conditions imply this property?

(iii) What about H^∞-functional calculus in this case?

A positive result on UMD Banach lattices

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Theorem

Let G be an abelian locally compact group. Let X be a UMD Banach lattice and let $\pi\colon G \to B(X)$ be a strongly continuous bounded representation. Let $\nu \in M(G)$ be a probability measure with BAR. Then

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Elements of proof.

- Consider $\lambda^X\colon G \to B(L^2(G;X))$ defined by $\lambda(t)f = f(\cdot - t)$ for any $f \in L^2(G:X)$ and any $t \in G$. Then we have

$$M_\nu^X := \int_G \lambda^X(t)\, d\nu(t) = \nu * \cdot : L^2(G;X) \to L^2(G;X).$$

A positive result on UMD Banach lattices (continued)

• By transference arguments, one shows that it suffices to consider the case when $\pi = \lambda^X$; the space X is replaced by $L^2(G; X)$ and $S(\pi, \nu)$ is replaced by M_ν^X.

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• Since X is a UMD Banach lattice, there exists a Hilbert space H, a UMD Banach space Y and $\theta \in (0, 1)$ such that

$$X = [Y, H]_\theta$$

in the sense of complex interpolation (Rubio de Francia, 1986).

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• Then $L^2(G; X) = [L^2(G; Y), L^2(G; H)]_\theta$ and M_ν^X interpolates betwen M_ν^H and M_ν^Y.

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- Since H is a Hilbert space, the operator M_ν^H behaves like M_ν: this is a Ritt operator with a bounded H^∞-functional calculus.

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- Since H is a Hilbert space, the operator M_ν^H behaves like M_ν: this is a Ritt operator with a bounded H^∞-functional calculus.
- By interpolation (Blunck, 2001), this implies that M_ν^X is a Ritt operator.
- Further interpolation arguments show that M_ν^X admits a bounded H^∞-functional calculus. (Here the UMD property of Y plays a role.)

A positive result on UMD Banach lattices (continued)

Corollary

Let $1 < p < \infty$, let (Ω, μ) be a measure space and let $T \in B(L^p(\Omega))$ be a positive contraction. Let $c = (c_k)_{k \geqslant 1}$ be a sequence satisfying BAR. Then

$$S = \sum_{k=0}^{\infty} c_k T^k$$

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The proof consists in applying Akcoglu's dilation Theorem and the previous Theorem on $G = \mathbb{Z}$.

K-convex Banach spaces

Let X be a Banach space. We say that X **contains** ℓ_n^1 **uniformly** if there is a constant $K \geqslant 1$ such that for any integer $n \geqslant 1$, there is an n-dimensional subspace $E_n \subset X$ such that $\ell_1^n \overset{K}{\simeq} E_n$. That is, for any $n \geqslant 1$, there exist $x_1, \ldots, x_n$ in X such that for any $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$,

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Pisier's Theorem, 1982

The following are equivalent.

(i) X does not contains ℓ_n^1's uniformly.

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UMD implies K-convex but there exist non reflexive K-convex spaces.

Symmetric measures

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A symmetric ν has BAR if and only if there exists $a \in (-1, 1)$ such that $\widehat{\nu}$ is actually valued in $[a, 1]$.

Symmetric measures

Let G be a locally compact abelian group and let $\nu \in M(G)$ be a probability. We say that ν is **symmetric** when $\nu(A) = \nu(-A)$ for any measurable $A \subset G$. The following is easy to check:

$$\begin{aligned} \nu \text{ is symmetric } &\Leftrightarrow M_\nu \colon L^2(G) \to L^2(G) \text{ is selfadjoint} \\ &\Leftrightarrow \widehat{\nu} \text{ is real valued} \\ &\Leftrightarrow \widehat{\nu} \text{ is valued in } [-1,1]. \end{aligned}$$

A symmetric ν has BAR if and only if there exists $a \in (-1,1)$ such that $\widehat{\nu}$ is actually valued in $[a,1]$.

If $\nu = \eta * \eta$ for some symmetric probability measure η, then ν is a symmetric probability measure with BAR, since $\widehat{\nu} = \widehat{\eta}^2$ is valued in $[0,1]$. We say that ν **is a square** in this case.

Main result

Theorem

Let ν be a symmetric probability measure on some locally compact abelian group G. Assume that ν is a square. Let X be a K-convex Banach space. Then for any bounded strongly continuous representation $\pi\colon G \to B(X)$,

$$S(\pi, \nu) = \int_G \pi(t)\, d\nu(t)$$

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The proof relies on some techniques introduced by Pisier, in particular the following result:

- *If Y is a K-convex Banach space, then there exists $C \geqslant 1$ such that for any integer $n \geqslant 1$ and for any commuting contractive projections in $B(Y)$, then*

$$\Big\| \sum_{j=1}^{n} (I_Y - P_j) \prod_{1\leqslant j\neq k\leqslant n} P_k \Big\| \leqslant C.$$

Main result (continued)

Let $\nu \in M(G)$ be a symmetric probability with BAR.
If $\nu \in L^1(G)$ (i.e. ν has a density), then one can get rid of the assumption that ν is a square in the preceding Theorem.

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This applies to any measure on a discrete group.
On $G = \mathbb{Z}$ this yields the following.

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On $G = \mathbb{Z}$ this yields the following.

Corollary

Let $c = (c_k)_{k\in\mathbb{Z}}$ be a nonnegative sequence with $\sum_k c_k = 1$. Assume that $\sum_k c_k e^{ik\theta} \geqslant 0$ for any $\theta \in \mathbb{R}$. Let X be a K-convex space and let $U \in B(X)$ be an invertible operator with $\sup_{k\in\mathbb{Z}} \|T^k\| < \infty$. Then

$$S = \sum_{k=-\infty}^{\infty} c_k U^k$$

is a Ritt operator.

Necessity of K-convexity

Theorem

Let G be a non-discrete locally compact abelian group. Let X be a Banach space and assume that X in not K-convex. Then there exists a symmetric probability measure $\nu \in M(G)$ such that $M^X_{\nu^2}$ is not a Ritt operator.

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Let G be a non-discrete locally compact abelian group. Let X be a Banach space and assume that X in not K-convex. Then there exists a symmetric probability measure $\nu \in M(G)$ such that $M_{\nu^2}^X$ is not a Ritt operator.

The proof relies on the existence of a symmetric probability measure $\nu \in M(G)$ such that $1 + \nu^2$ is not invertible in $M(G)$ (a refinement of the Wiener-Pitt Theorem).