Fredholmness of Some Toeplitz Operators

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Definition (Fredholm operator)

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. T is said to be Fredholm operator if the range of T is closed, $\dim\mathrm{Ker}T$ and $\dim\mathrm{Ker}T^*$ are finite.

Preliminaries

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Theorem (Atkinson's characterization)

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. Then T is Fredholm operator if and only if $T + \mathcal{K}(H)$ is invertible in the quotient algebra $\mathcal{B}(H)/\mathcal{K}(H)$, where $\mathcal{K}(H)$ is the ideal of all compact operators on H.

A Banach algebra is a complex normed algebra $\mathcal{A}$ which is complete (as a topological space) and satisfies

$$||ST|| \leq ||S|| \, ||T|| \text{ for all } S, T \in \mathcal{A}.$$

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A C^*-algebra is a Banach algebra $\mathcal{A}$ with conjugate-linear involution $*$ which is an anti-isomorphism, that is, for all $S, T \in \mathcal{A}$ and λ in $\mathbb{C}$

$$(\lambda S + T)^* = \bar{\lambda} S^* + T^*,$$

$$(ST)^* = T^* S^*,$$

$$(S^*)^* = S$$

and additional norm condition

$$||S^* S|| = ||S||^2 \text{ for all } S \in \mathcal{A}.$$

If $\mathcal{B}$ is a subset of a C^*-algebra $\mathcal{A}$, we set $\mathcal{B}^* = \{T^* : T \in \mathcal{B}\}$ and if $\mathcal{B}^* = \mathcal{B}$ we say $\mathcal{B}$ is self-adjoint. A closed self-adjoint subalgebra $\mathcal{B}$ of $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{A}$.

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*Any C^*-algebra is isometrically *-isomorphic to a C^*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.*

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If $\mathcal{A}$ is a C^*-algebra, then its commutator ideal $\mathcal{I}$ is the smallest (norm) closed, two-sided ideal of $\mathcal{A}$ containing $\{AB - BA : A, B \in \mathcal{A}\}$.

C^*-algebras generated by a system of unilateral weighted shifts

Let n be a fixed positive integer.

$I = (i_1, \ldots, i_n)$ be a multi-index of integers.

$I \geq 0$: $i_j \geq 0$ for all $j = 1, \ldots, n$

$|I| = |i_1 + \ldots + i_n|$

For $I \geq 0$

$$z^I = z_1^{i_1} \ldots z_n^{i_n},$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

Let $\varepsilon_k = (\delta_{1k}, \ldots, \delta_{nk})$ be an another multi-index, where δ_{ij} is the Kronocker symbol. For the multi-index I

$$I \mp \varepsilon_k = (i_1, \ldots, i_k \mp 1, \ldots, i_n)$$

Let $\{e_I\}$ be an orthonormal basis of a complex Hilbert space H and let $\{w_{I,j} : j = 1, \ldots, n\}$ be a bounded set of complex numbers such that

$$w_{I,k} w_{I+\varepsilon_k,t} = w_{I,t} w_{I+\varepsilon_t,k}$$

for all I and $1 \leq k, t \leq n$.

Definition (Jewell, Lubin)

A system of unilateral weighted shifts is a family of n-operators $A = \{A_1, \ldots, A_n\}$ on H such that

$$A_j e_I = w_{I,j} e_{I+\varepsilon_j}, \ I \geq 0, \ j = 1, \ldots, n.$$

With aid of the positive $\{w_{I,j} : j = 1, \ldots, n\}$ we define a set $\{\beta_I\}_{I \geq 0}$ such that

$$\beta_{I+\varepsilon_j} = w_{I,j}\beta_I; \ \ \beta_0 = 1.$$

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Then, the space

$$H^2(\beta) = \left\{ f(z) = \sum_{I \geq 0} f_I z^I : \sum_{I \geq 0} |f_I|^2 \beta_I^2 < \infty \right\}.$$

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is a Hilbert space with the inner product

$$\langle f, g \rangle = \sum_{I \geq 0} f_I \overline{g_I} \beta_I^2$$

and $\{\frac{z^I}{\beta_I}\}_{I \geq 0}$ is an orthonormal basis for $H^2(\beta)$ (Jewell-Lubin).

Consider such $\{\beta_I\}_{I\geq 0}$ that the multi-variable moment problem

$$\beta_I^2 = \int_{[0,1]^n} r_1^{2i_1} r_2^{2i_2} \ldots r_n^{2i_n} d\nu(r_1, r_2, \ldots, r_n)$$

has a solution for these β_I^2's, that is, there exists a positive Borel measure ν defined on $[0,1]^n$ for these β_I^2's satisfying above equality.

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Let Ω denote the family of the systems A such that β_I^2's corresponding to the system A satisfying above property (i.e., the multi-variable moment problem has a solution for β_I^2's) and let ν_A denote the measure corresponding to the system A.

$L^2(\bar{\mathbb{D}}^n, \mu)$ $(= L^2(\mu))$: the space of complex-valued functions on $\bar{\mathbb{D}}^n$ which are Lebesgue measurable and square-integrable with respect to the measure μ. Here μ is given on $\bar{\mathbb{D}}^n$ by

$$d\mu = \frac{1}{(2\pi)^n} d\nu(r_1, r_2, \ldots, r_n) d\theta_1 d\theta_2 \ldots d\theta_n \ \ (0 < \theta_i \leq 2\pi)$$

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If $A \in \Omega$, then $\left\{ \frac{1}{\sqrt{(2\pi)^n}} \frac{z^I}{\beta_I} \right\}_{I \geq 0}$ is an orthonormal system in $L^2(\mu)$.

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If $A \in \Omega$, then $\left\{ \frac{1}{\sqrt{(2\pi)^n}} \frac{z^l}{\beta_l} \right\}_{l \geq 0}$ is an orthonormal system in $L^2(\mu)$.

Let $H^2(\bar{\mathbb{D}}^n, \mu)$ $(= H^2(\mu))$ be the subspace generated by the orthonormal system $\left\{ \frac{1}{\sqrt{(2\pi)^n}} \frac{z^l}{\beta_l} \right\}_{l \geq 0} \in L^2(\mu)$.

Functional Model

If $A \in \Omega$, then the system A of unilateral weighted shifts on H and the system $A_z (= \{A_{z_1}, \ldots, A_{z_n}\})$ of the multiplication operators A_{z_i} on $H^2(\mu)$ by the independent variables z_j's, $i = 1, 2, \ldots, n$ are unitarily equivalent.

For simplicity, we consider $n = 2$. Let be

$$S_1 = \{(r_1, r_2) \in [0,1] \times [0,1] : r_1^2 + r_2^2 \leq 1\}$$

$$\widetilde{S}_1 = \{(r_1, r_2) \in [0,1] \times [0,1] : r_1^2 + r_2^2 = 1\}.$$

Let Ω_1 be the subset of Ω defined by

$$\Omega_1 = \{A \in \Omega : supp\nu_A \subset S_1,\ \nu_A(U(a)) > 0 \text{ for arbitrary}$$

$$\text{neighborhood } U(a) \text{ of each point } a \in \widetilde{S}_1\}$$

Theorem

If $A \in \Omega_1$, then $H^2(\mu)$ is a functional Hilbert space.

Theorem (Ergezen, Sadik)

Let $A \in \Omega$. A necessary and sufficient condition for the operator algebra generated by the system A to be isometrically isomorphic to the ball algebra is that A is in Ω_1.

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Theorem (K., Sadik)

Let $A \in \Omega$. If the algebra generated by the system A is the ball algebra, then the commutator ideal of the C^*- algebra $C^*(A)$ generated by the system A is the ideal of all compact operators $\mathcal{K}$ and

$$C^*(A) = \{T_\psi + K : \psi \in C(supp\mu), \;\; K \in \mathcal{K}\}.$$

The quotient $C^*(A)/\mathcal{K}$ is $*$-isomorphic to $C(S^3)$, where S^3 is the unit sphere.

Definition

If P denotes the orthogonal projection from $L^2(\mu)$ onto $H^2(\mu)$, then for $\psi \in C(supp\mu)$ the Toeplitz operator T_ψ on $H^2(\mu)$ with continuous symbol ψ is defined by

$$T_\psi f = P(\psi f)$$

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Corollary

Let $\psi \in C(supp\mu)$. *Then The Toeplitz operator* $T_\psi \in C^*(A)$ *is Fredholm if and only if* $\psi(z) \neq 0$ *for all* $z \in S^3$.

Let Ω_2 be the subset of Ω defined by

$$\Omega_2 = \{A \in \Omega : \nu_A(U(1,1)) > 0 \text{ for arbitrary neighborhood } U(1,1) \text{ of the point } (1,1) \in [0,1]^2\}.$$

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If $A \in \Omega_2$, then $H^2(\mu)$ is a functional Hilbert space.

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Let $A \in \Omega_2$. Then, the commutator ideal $\mathcal{J}$ of $C^*(A)$ properly contains the ideal $\mathcal{K}$ of compact operators on $H^2(\mathbb{D}^2)$. The quotient space $\mathcal{J}/\mathcal{K}(H^2(\mathbb{D}^2))$ is isometrically isomorphic to $C(\mathbb{T} \times \{0,1\}) \otimes \mathcal{K}(H^2(\mathbb{D}))$, where $\mathbb{T}$ is the unit circle and $\{0,1\}$ is the two-point space.

We assume that the measure ν_A has of the form

$$\nu_A(r_1, r_2) = \nu_1(r_1)\nu_2(r_2),$$

where both measures ν_1 and ν_2 are defined on $[0, 1]$ and satisfying $\nu_i(a, 1] > 0$, $i = 1, 2$ for all $0 < a < 1$. The measure μ_A is then written as

$$\mu_A = \frac{1}{(2\pi)^2}\nu_1\nu_2 d\theta_1 d\theta_2$$

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Theorem (K., Sadik)

Let $\psi \in C(supp\mu)$. Then a necessary and sufficient condition for T_ψ to be Fredholm is that ψ does not vanish in $\mathbb{T}^2$ and $\psi|_{\mathbb{T}^2}$ is homotopic to a constant.

An application for the unit ball case

π: the quotient homomorphism from $B(H)$ to $B(H)/\mathcal{K}(H)$

L: a subalgebra of $B(H)$ such that the image $\pi(L)$ is a commutative subalgebra of $B(H)/\mathcal{K}(H)$

S: an automorphism in the algebra $\pi(L)$ such that $S\pi(B) = \pi(B')S$, where $B, B' \in L$, that is, if $B \in L$, then $SBS^{-1} = B' + K$, $B' \in L$ and $K \in \mathcal{K}(H)$.

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Consider the operator

$$T = B_1 + B_2 S + K,$$

where $B_1, B_2 \in L$ ve $K \in \mathcal{K}$.

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Theorem (K., Sadik)

If the operator $\pi(B_1)\pi(B_1') - \pi(B_2)\pi(B_2')$ has an inverse in $\pi(L)$, then $T = B_1 + B_2 S + K$ is Fredholm.

Take $A \in \Omega_1$. Let $L = C^*(A)$. Consider the operator

$$T = T_{\psi_1} + T_{\psi_2} S + K,$$

where T_{ψ_1} ve T_{ψ_2} are Toeplitz operators in $C^*(A)$ with the symbols $\psi_1, \psi_2 \in C(supp\mu)$, respectively and $Sf(w_1, w_2) = f(w_2, w_1)$ for all $f \in H^2(\mu)$.

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The equation $Tf = \varphi$ is written of the form

$$\int_{B^4} K(z,w)\psi_1(w_1,w_2)f(w_1,w_2)d\mu(w_1,w_2)+$$

$$\int_{B^4} K(z,w)\psi_2(w_1,w_2)f(w_2,w_1)d\mu(w_1,w_2) + (Kf)(z_1,z_2) = \varphi(z_1,z_2)$$

Theorem (K., Sadik)

If $\psi_1(z_1, z_2)\psi_1(z_2, z_1) - \psi_2(z_1, z_2)\psi_2(z_2, z_1)$ *does not vanish in* S^3*, then all of Noether's theorems is true for the equation* $Tf = \varphi$*.*

Theorem (K., Sadik)

If $\psi_1(z_1, z_2)\psi_1(z_2, z_1) - \psi_2(z_1, z_2)\psi_2(z_2, z_1)$ *does not vanish in* S^3*, then all of Noether's theorems is true for the equation* $Tf = \varphi$*.In particular, if we take* $w_{I,1} = \sqrt{\frac{m+1}{2+m+n}}$, $w_{I,2} = \sqrt{\frac{n+1}{2+m+n}}$ *and* $Sf(w_1, w_2) = f(w_2, w_1)$ *then the equation above has the form*

$$\int_{S^3} \frac{\psi_1(w_1, w_2)f(w_1, w_2)}{(1 - z_1\bar{w}_1 - z_2\bar{w}_2)^2} ds + \int_{S^3} \frac{\psi_2(w_1, w_2)f(w_2, w_1)}{(1 - z_1\bar{w}_1 - z_2\bar{w}_2)^2} ds +$$

$$+(Kf)(z_1, z_2) = \varphi(z_1, z_2),$$

where ds *is the surface measure in* S^3*.*

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