Fredholmness of Some Toeplitz Operators

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Preliminaries

Definition (Fredholm operator)
Let $H$ be a Hilbert space and let $T \in \mathcal{B}(H)$. $T$ is said to be Fredholm operator if the range of $T$ is closed, $\dim \ker T$ and $\dim \ker T^*$ are finite.
### Preliminaries

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Let $H$ be a Hilbert space and let $T \in B(H)$. $T$ is said to be **Fredholm operator** if the range of $T$ is closed, $\dim \ker T$ and $\dim \ker T^*$ are finite.

**Theorem (Atkinson’s characterization)**

Let $H$ be a Hilbert space and let $T \in B(H)$. Then $T$ is Fredholm operator if and only if $T + \mathcal{K}(H)$ is invertible in the quotient algebra $B(H)/\mathcal{K}(H)$, where $\mathcal{K}(H)$ is the ideal of all compact operators on $H$. 

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A Banach algebra is a complex normed algebra $A$ which is complete (as a topological space) and satisfies

$$\|ST\| \leq \|S\| \|T\| \text{ for all } S, T \in A.$$
A **Banach algebra** is a complex normed algebra $\mathcal{A}$ which is complete (as a topological space) and satisfies

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A **$C^*$-algebra** is a Banach algebra $\mathcal{A}$ with conjugate-linear involution $\ast$ which is an anti-isomorphism, that is, for all $S, T \in \mathcal{A}$ and $\lambda$ in $\mathbb{C}$

$$(\lambda S + T)^* = \overline{\lambda} S^* + T^*,$$

$$(ST)^* = T^* S^*,$$

$$(S^*)^* = S$$

and additional norm condition

$$\|S^* S\| = \|S\|^2 \text{ for all } S \in \mathcal{A}.$$
If $\mathcal{B}$ is a subset of a $C^*$-algebra $\mathcal{A}$, we set $\mathcal{B}^* = \{ T^* : T \in \mathcal{B} \}$ and if $\mathcal{B}^* = \mathcal{B}$ we say $\mathcal{B}$ is self-adjoint. A closed self-adjoint subalgebra $\mathcal{B}$ of $\mathcal{A}$ is a $C^*$-subalgebra of $\mathcal{A}$. 

Theorem (Gelfand-Naimark Theorem) Any $C^*$-algebra is isometrically *-isomorphic to a $C^*$-subalgebra of $\mathcal{B}(H)$ for some Hilbert space $H$. 

For every subset $\mathcal{C}$ of $\mathcal{A}$ there is a smallest $C^*$-subalgebra of $\mathcal{A}$ containing $\mathcal{C}$, called the $C^*$-subalgebra generated by $\mathcal{C}$. 

If $\mathcal{A}$ is a $C^*$-algebra, then its commutator ideal $I$ is the smallest (norm) closed, two-sided ideal of $\mathcal{A}$ containing $\{ AB - BA : A, B \in \mathcal{A} \}$. 

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If $B$ is a subset of a $C^*$-algebra $A$, we set $B^* = \{ T^* : T \in B \}$ and if $B^* = B$ we say $B$ is self-adjoint. A closed self-adjoint subalgebra $B$ of $A$ is a $C^*$-subalgebra of $A$.

**Theorem (Gelfand-Naimark Theorem)**

Any $C^*$-algebra is isometrically $*$-isomorphic to a $C^*$-subalgebra of $B(H)$ for some Hilbert space $H$. 
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If $A$ is a $C^*$-algebra, then its **commutator ideal** $\mathcal{I}$ is the smallest (norm) closed, two-sided ideal of $A$ containing $\{ AB - BA : A, B \in A \}$.
C*-algebras generated by a system of unilateral weighted shifts

Let $n$ be a fixed positive integer.

$I = (i_1, \ldots, i_n)$ be a multi-index of integers.

$I \geq 0$: $i_j \geq 0$ for all $j = 1, \ldots, n$

$|I| = |i_1 + \ldots + i_n|$

For $I \geq 0$

$$z^I = z_1^{i_1} \cdots z_n^{i_n},$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

Let $\varepsilon_k = (\delta_{1k}, \ldots, \delta_{nk})$ be another multi-index, where $\delta_{ij}$ is the Kronoccker symbol. For the multi-index $I$

$$I \mp \varepsilon_k = (i_1, \ldots, i_k \mp 1, \ldots, i_n)$$
Let \( \{e_I\} \) be an orthonormal basis of a complex Hilbert space \( H \) and let \( \{w_I,j : j = 1, \ldots, n\} \) be a bounded set of complex numbers such that

\[
 w_{I,k} w_{I+\varepsilon_k,t} = w_{I,t} w_{I+\varepsilon_t,k}
\]

for all \( I \) and \( 1 \leq k, t \leq n \).

**Definition (Jewell, Lubin)**

A system of unilateral weighted shifts is a family of \( n \)-operators \( A = \{A_1, \ldots, A_n\} \) on \( H \) such that

\[
 A_j e_I = w_{I,j} e_{I+\varepsilon_j}, \quad I \geq 0, \; j = 1, \ldots, n.
\]
With aid of the positive \{w_{I,j} : j = 1, \ldots, n\} we define a set \{\beta_I\}_{I \geq 0} such that

\[
\beta_{I+\varepsilon_j} = w_{I,j} \beta_I; \quad \beta_0 = 1.
\]
With aid of the positive \( \{w_{l,j} : j = 1, \ldots, n\} \) we define a set \( \{\beta_l\}_{l \geq 0} \) such that
\[
\beta_{l+\epsilon_j} = w_{l,j} \beta_l; \quad \beta_0 = 1.
\]

Then, the space
\[
H^2(\beta) = \left\{ f(z) = \sum_{l \geq 0} f_l z^l : \sum_{l \geq 0} |f_l|^2 \beta_{2l}^2 < \infty \right\}.
\]
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Then, the space
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H^2(\beta) = \left\{ f(z) = \sum_{I \geq 0} f_I z^I : \sum_{I \geq 0} |f_I|^2 \beta^2_I < \infty \right\}
\]
is a Hilbert space with the inner product
\[
\langle f, g \rangle = \sum_{I \geq 0} f_I \overline{g_I} \beta^2_I
\]
and \( \{\frac{z^I}{\beta_I}\}_{I \geq 0} \) is an orthonormal basis for \( H^2(\beta) \) (Jewell-Lubin).
Consider such \( \{\beta_I\}_{I \geq 0} \) that the multi-variable moment problem

\[
\beta^2_I = \int_{[0,1]^n} r_1^{2i_1} r_2^{2i_2} \ldots r_n^{2i_n} d\nu(r_1, r_2, \ldots, r_n)
\]

has a solution for these \( \beta^2_I \)'s, that is, there exists a positive Borel measure \( \nu \) defined on \([0,1]^n\) for these \( \beta^2_I \)'s satisfying above equality.
Consider such \( \{\beta_i\}_{i \geq 0} \) that the multi-variable moment problem

\[
\beta_i^2 = \int_{[0,1]^n} r_1^{2i_1} r_2^{2i_2} \cdots r_n^{2i_n} d\nu(r_1, r_2, \ldots, r_n)
\]

has a solution for these \( \beta_i^2 \)'s, that is, there exists a positive Borel measure \( \nu \) defined on \([0,1]^n\) for these \( \beta_i^2 \)'s satisfying above equality.

Let \( \Omega \) denote the family of the systems \( A \) such that \( \beta_i^2 \)'s corresponding to the system \( A \) satisfying above property (i.e., the multi-variable moment problem has a solution for \( \beta_i^2 \)'s)
Consider such \( \{\beta_i\}_{i \geq 0} \) that the multi-variable moment problem

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has a solution for these \( \beta_i^2 \)'s, that is, there exists a positive Borel measure \( \nu \) defined on \([0,1]^n\) for these \( \beta_i^2 \)'s satisfying above equality.

Let \( \Omega \) denote the family of the systems \( A \) such that \( \beta_i^2 \)'s corresponding to the system \( A \) satisfying above property (i.e., the multi-variable moment problem has a solution for \( \beta_i^2 \)'s) and let \( \nu_A \) denote the measure corresponding to the system \( A \).
$L^2(\mathbb{D}^n, \mu) \ (= L^2(\mu))$ : the space of complex-valued functions on $\mathbb{D}^n$ which are Lebesgue measurable and square-integrable with respect to the measure $\mu$. Here $\mu$ is given on $\mathbb{D}^n$ by

$$d\mu = \frac{1}{(2\pi)^n} d\nu(r_1, r_2, \ldots, r_n) d\theta_1 d\theta_2 \ldots d\theta_n \ (0 < \theta_i \leq 2\pi)$$
$L^2(\bar{D}^n, \mu) \ (= L^2(\mu))$ : the space of complex-valued functions on $\bar{D}^n$ which are Lebesgue measurable and square-integrable with respect to the measure $\mu$. Here $\mu$ is given on $\bar{D}^n$ by

$$d\mu = \frac{1}{(2\pi)^n} d\nu(r_1, r_2, \ldots, r_n) d\theta_1 d\theta_2 \ldots d\theta_n \ (0 < \theta_i \leq 2\pi)$$

If $A \in \Omega$, then $\left\{ \frac{1}{\sqrt{(2\pi)^n}} \frac{z^I}{\beta_I} \right\}_{I \geq 0}$ is an orthonormal system in $L^2(\mu)$. 
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If $A \in \Omega$, then $\left\{ \frac{1}{\sqrt{(2\pi)^n}} \frac{z^l}{\beta_l} \right\}_{l \geq 0}$ is an orthonormal system in $L^2(\mu)$.

Let $H^2(\mathbb{D}^n, \mu) \ (= H^2(\mu))$ be the subspace generated by the orthonormal system $\left\{ \frac{1}{\sqrt{(2\pi)^n}} \frac{z^l}{\beta_l} \right\}_{l \geq 0} \in L^2(\mu)$. 
If $A \in \Omega$, then the system $A$ of unilateral weighted shifts on $H$ and the system $A_z(= \{A_{z_1}, \ldots, A_{z_n}\})$ of the multiplication operators $A_{z_i}$ on $H^2(\mu)$ by the independent variables $z_j$’s, $i = 1, 2, \ldots, n$ are unitarily equivalent.
For simplicity, we consider $n = 2$. Let be

$$S_1 = \{(r_1, r_2) \in [0, 1] \times [0, 1] : r_1^2 + r_2^2 \leq 1\}$$

$$\tilde{S}_1 = \{(r_1, r_2) \in [0, 1] \times [0, 1] : r_1^2 + r_2^2 = 1\}.$$ 

Let $\Omega_1$ be the subset of $\Omega$ defined by

$$\Omega_1 = \{A \in \Omega : \text{supp} \nu_A \subset S_1, \nu_A(U(a)) > 0 \text{ for arbitrary neighborhood } U(a) \text{ of each point } a \in \tilde{S}_1\}$$

**Theorem**

*If $A \in \Omega_1$, then $H^2(\mu)$ is a functional Hilbert space.*
Theorem (Ergezen, Sadik)

Let $A \in \Omega$. A necessary and sufficient condition for the operator algebra generated by the system $A$ to be isometrically isomorphic to the ball algebra is that $A$ is in $\Omega_1$. 

Theorem (K., Sadik)

Let $A \in \Omega$. If the algebra generated by the system $A$ is the ball algebra, then the commutator ideal of the $C^*$-algebra $C^*(A)$ generated by the system $A$ is the ideal of all compact operators $K$ and $C^*(A) = \{T\psi + K : \psi \in C(supp\mu), K \in K\}$.

The quotient $C^*(A)/K$ is $*$-isomorphic to $C(S^3)$, where $S^3$ is the unit sphere.
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$$C^*(A) = \{ T_\psi + K : \psi \in C(\text{supp}\mu), \ K \in K \}.$$ 

The quotient $C^*(A)/K$ is $*$-isomorphic to $C(S^3)$, where $S^3$ is the unit sphere.
Definition

If $P$ denotes the orthogonal projection from $L^2(\mu)$ onto $H^2(\mu)$, then for $\psi \in C(supp\mu)$ the Toeplitz operator $T_\psi$ on $H^2(\mu)$ with continuous symbol $\psi$ is defined by

$$T_\psi f = P(\psi f)$$

for $f \in H^2(\mu)$. 

Corollary

Let $\psi \in C(supp\mu)$. Then the Toeplitz operator $T_\psi$ is Fredholm if and only if $\psi(z) \neq 0$ for all $z \in \mathbb{S}^3$. 

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Let $\Omega_2$ be the subset of $\Omega$ defined by

$$\Omega_2 = \{ A \in \Omega : \nu_A(U(1,1)) > 0 \text{ for arbitrary neighborhood } U(1,1) \text{ of the point } (1,1) \in [0,1]^2 \}.$$ 

**Theorem**

If $A \in \Omega_2$, then $H^2(\mu)$ is a functional Hilbert space.
Theorem (Ergezen, Sadik)

Let \( A \in \Omega \). A necessary and sufficient condition for the operator algebra generated by the system \( A \) to be isometrically isomorphic to the polydisc algebra is that \( A \) is in \( \Omega_2 \).
Theorem (Ergezen, Sadik)

Let $A \in \Omega$. A necessary and sufficient condition for the operator algebra generated by the system $A$ to be isometrically isomorphic to the polydisc algebra is that $A$ is in $\Omega_2$.

Theorem (K., Sadik)

Let $A \in \Omega_2$. Then, the commutator ideal $\mathcal{J}$ of $C^*(A)$ properly contains the ideal $\mathcal{K}$ of compact operators on $H^2(D^2)$. The quotient space $\mathcal{J}/\mathcal{K}(H^2(D^2))$ is isometrically isomorphic to $C(T \times \{0, 1\}) \otimes \mathcal{K}(H^2(D))$, where $T$ is the unit circle and $\{0, 1\}$ is the two-point space.
We assume that the measure $\nu_A$ has of the form

$$\nu_A(r_1, r_2) = \nu_1(r_1)\nu_2(r_2),$$

where both measures $\nu_1$ and $\nu_2$ are defined on $[0, 1]$ and satisfying $\nu_i(a, 1] > 0$, $i = 1, 2$ for all $0 < a < 1$. The measure $\mu_A$ is then written as

$$\mu_A = \frac{1}{(2\pi)^2} \nu_1 \nu_2 d\theta_1 d\theta_2.$$

Theorem (K., Sadik)

Let $\psi \in C(\text{supp } \mu_A)$. Then a necessary and sufficient condition for $T\psi$ to be Fredholm is that $\psi$ does not vanish in $T$ and $\psi|_{T}$ is homotopic to a constant.
We assume that the measure $\nu_A$ has of the form

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\[\text{Theorem (K., Sadik)}\]

Let $\psi \in C(\text{supp}\mu)$. Then a necessary and sufficient condition for $T_\psi$ to be Fredholm is that $\psi$ does not vanish in $\mathbb{T}^2$ and $\psi|_{\mathbb{T}^2}$ is homotopic to a constant.
An application for the unit ball case

\( \pi \): the quotient homomorphism from \( B(H) \) to \( B(H)/\mathcal{K}(H) \)

\( L \): a subalgebra of \( B(H) \) such that the image \( \pi(L) \) is a commutative subalgebra of \( B(H)/\mathcal{K}(H) \)

\( S \): an automorphism in the algebra \( \pi(L) \) such that \( S\pi(B) = \pi(B')S \), where \( B, B' \in L \), that is, if \( B \in L \), then \( SBS^{-1} = B' + K \), \( B' \in L \) and \( K \in \mathcal{K}(H) \).
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Consider the operator

\[
T = B_1 + B_2S + K,
\]

where \( B_1, B_2 \in L \) ve \( K \in \mathcal{K} \).
An application for the unit ball case

\(\pi: \) the quotient homomorphism from \(B(H)\) to \(B(H)/\mathcal{K}(H)\)

\(L: \) a subalgebra of \(B(H)\) such that the image \(\pi(L)\) is a commutative subalgebra of \(B(H)/\mathcal{K}(H)\)

\(S: \) an automorphism in the algebra \(\pi(L)\) such that \(S\pi(B) = \pi(B')S\), where \(B, B' \in L\), that is, if \(B \in L\), then \(SBS^{-1} = B' + K, B' \in L\) and \(K \in \mathcal{K}(H)\).

Consider the operator

\[ T = B_1 + B_2S + K, \]

where \(B_1, B_2 \in L\) and \(K \in \mathcal{K}\).

**Theorem (K., Sadik)**

*If the operator \(\pi(B_1)\pi(B'_1) - \pi(B_2)\pi(B'_2)\) has an inverse in \(\pi(L)\), then \(T = B_1 + B_2S + K\) is Fredholm.*
Take $A \in \Omega_1$. Let $L = C^*(A)$. Consider the operator

$$ T = T_{\psi_1} + T_{\psi_2} S + K, $$

where $T_{\psi_1}$ ve $T_{\psi_2}$ are Toeplitz operators in $C^*(A)$ with the symbols $\psi_1, \psi_2 \in C(supp \mu)$, respectively and $Sf(w_1, w_2) = f(w_2, w_1)$ for all $f \in H^2(\mu)$. 
Take $A \in \Omega_1$. Let $L = C^*(A)$. Consider the operator

$$T = T_{\psi_1} + T_{\psi_2} S + K,$$

where $T_{\psi_1}$ ve $T_{\psi_2}$ are Toeplitz operators in $C^*(A)$ with the symbols $\psi_1, \psi_2 \in C(supp\mu)$, respectively and $Sf(w_1, w_2) = f(w_2, w_1)$ for all $f \in H^2(\mu)$. The equation $Tf = \varphi$ is written of the form

$$\int_{B^4} K(z, w)\psi_1(w_1, w_2)f(w_1, w_2)d\mu(w_1, w_2) +$$

$$\int_{B^4} K(z, w)\psi_2(w_1, w_2)f(w_2, w_1)d\mu(w_1, w_2) + (Kf)(z_1, z_2) = \varphi(z_1, z_2)$$
Theorem (K., Sadik)

If \( \psi_1(z_1, z_2) \psi_1(z_2, z_1) - \psi_2(z_1, z_2) \psi_2(z_2, z_1) \) does not vanish in \( S^3 \), then all of Noether’s theorems is true for the equation \( Tf = \varphi \).
Theorem (K., Sadik)

If \( \psi_1(z_1, z_2)\psi_1(z_2, z_1) - \psi_2(z_1, z_2)\psi_2(z_2, z_1) \) does not vanish in \( S^3 \), then all of Noether's theorems is true for the equation \( Tf = \varphi \).

In particular, if we take \( w_{l,1} = \sqrt{\frac{m+1}{2+m+n}} \), \( w_{l,2} = \sqrt{\frac{n+1}{2+m+n}} \) and \( Sf(w_1, w_2) = f(w_2, w_1) \) then the equation above has the form

\[
\int_{S^3} \frac{\psi_1(w_1, w_2)f(w_1, w_2)}{(1 - z_1 \bar{w}_1 - z_2 \bar{w}_2)^2} ds + \int_{S^3} \frac{\psi_2(w_1, w_2)f(w_2, w_1)}{(1 - z_1 \bar{w}_1 - z_2 \bar{w}_2)^2} ds + (Kf)(z_1, z_2) = \varphi(z_1, z_2),
\]

where \( ds \) is the surface measure in \( S^3 \).


References


Koca, B.B., 2016, Fredholmness of Toeplitz operators on generalized Hardy spaces over the polydisc”, *Arch. Math.*, vol.107, pp.265-270.