Xu-Yung-Zwart theorem and C_0-groups generated by operators with non-basis family of eigenvectors

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IWOTA 2017,
Technische Universität Chemnitz, Germany, 14-18 August 2017

A one-parameter family $\{T(t)\}_{t\geq 0} : \mathbb{R}_+ \mapsto [X]$ – C_0-semigroup if:

1. $T(t)T(s) = T(t+s),\ t, s \geq 0;$
2. $T(0) = I;$
3. $\lim\limits_{t\downarrow 0} \|T(t)x - x\| = 0,\ x \in X.$

C_0-semigroups play important role in

- Operator theory
- Theory of PDE's
- Infinite-dimensional linear systems theory.

Generator of C_0-semigroup $\{T(t)\}_{t\geq 0}$ – operator $A : X \supset D(A) \mapsto X$,

which acts by the formula $Ax = \lim\limits_{t\downarrow 0} \frac{T(t)x-x}{t}$, $x \in D(A)$, with

$$D(A) = \left\{ x \in X : \ \exists \lim_{t\downarrow 0} \frac{T(t)x-x}{t} \right\}.$$

Central problems of C_0-semigroup theory are

1. To examine whether a concrete operator A is the generator of C_0-semigroup, and
2. To obtain the representation of this C_0-semigroup.

Theorem (E. Hille, K. Yosida, R. Phillips, W. Feller, I. Miyadera)

The operator $A : X \supset D(A) \mapsto X$ is the infinitesimal generator of C_0-semigroup $\{T(t)\}_{t\geq 0}$ satisfying $\|T(t)\| \leq Me^{\omega t}$ if and only if

1. $D(A)$ is dense, A is closed, and
2. $(\omega, +\infty) \subseteq \rho(A)$ and $\forall \lambda > \omega$, $\forall n \in \mathbb{N}$ we have

$$\left\|(\lambda I - A)^{-n}\right\| \leq \frac{M}{(\lambda - \omega)^n}.$$

Complexity of conditions 1 and 2. Lumer-Phillips theorem is much more useful but it covers only the case of contraction semigroups ($M = 1$, $\omega = 0$).

Riesz bases play important role in an infinite-dimensional linear systems theory and signal processing.

Techniques associated with Riesz bases are applied in the study of

1. Stability
2. Controllability
3. Stabilization
4. Observability
5. Spectral assignment
6. Asymptotic properties

of various infinite-dimensional linear systems.

In particular, R. Rabah, G. M. Sklyar, A. V. Rezounenko,

K. V. Sklyar, P. Barkhaev, P. Polak (University of Szczecin, Poland & V. N. Karazin Kharkiv University, Ukraine, 2003-2016) studied all these properties for linear delay systems of neutral type.

Theorem (G.Q. Xu & S.P. Yung, JDE, 2005, H. Zwart, JDE, 2010)

Let A – generator of the C_0-group in H, with simple eigenv. $\{\lambda_n\}_1^\infty$ and the corresp. (normalized) eigenvect. $\{\phi_n\}_1^\infty$. If $\overline{Lin}\{\phi_n\}_1^\infty = H$ and

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0, \tag{1}$$

then $\{\phi_n\}_1^\infty$ forms a Riesz basis of H.

Theorem (H. Zwart, JDE, 2010)

Let A be the generator of the C_0-group in H with eigenvalues $\{\lambda_n\}_1^\infty$. If the system of generalized eigenvectors is dense and

$$\{\lambda_n\}_1^\infty = \bigcup_{j=1}^{K} \{\lambda_{n,j}\}_{n=1}^\infty, \text{ where } \inf_{n\neq m} |\lambda_{n,k} - \lambda_{m,k}| > 0, \ k = 1, \ldots, K, \tag{2}$$

then $\exists$ spectral projections $\{P_n\}_1^\infty$ of A such that $\{P_nH\}_1^\infty$ is a Riesz basis of A-invariant subspaces of H and $\max_n \dim P_nH \leq K$.

What if the eigenvalues do not satisfy the condition (2)?

In particular:

1. Is it possible to construct the generator A of the C_0-group with purely imaginary eigenvalues, which don't satisfy (2), and dense family of eigenvectors, which don't form a Schauder basis?
2. When the Cauchy problem with such an operator A is well/ill–posed?

In a joint work with Prof. Grigory Sklyar we obtain the following

Answers:

1. Yes, and we construct the class of generators of C_0-groups with these preassigned properties.
2. The well-posedness of the Cauchy problem with such an operator A essentially depends on the asymptotic behaviour of its eigenvalues $\{\lambda_n\}_1^\infty$ at $i\infty$. We found conditions on the asymptotic behaviour of $\{\lambda_n\}_1^\infty$ under which the corresponding Cauchy problem is well/ill–posed.

To obtain these results we

- Introduce and study special classes of Hilbert spaces $H_k(\{e_n\})$, $k \in \mathbb{N}$. Space $H_k(\{e_n\})$ depend on an arbitrary separable Hilbert space H and a chosen Riesz basis $\{e_n\}_1^\infty$ of H.
- Prove that $\{e_n\}_1^\infty$ is dense and minimal in $H_k(\{e_n\})$ but not uniformly minimal, hence do not form a Schauder basis.
- Consider the classes $\mathcal{S}_k$, $k \in \mathbb{N}$, of increasing sequences $\{f(n)\}_{n=1}^\infty \subset \mathbb{R}$ satisfying

$$\left\{n^j \Delta^j f(n)\right\}_{n=1}^\infty \in \ell_\infty$$

for $1 \le j \le k$, where Δ is a difference operator.

Example (For every $k \in \mathbb{N}$:)

1. $\{\ln n\}_{n=1}^\infty \in \mathcal{S}_k$, $\{\ln\ln(n+1)\}_{n=1}^\infty \in \mathcal{S}_k$,
2. $\left\{\ln\ln\sqrt{n+1}\right\}_{n=1}^\infty \in \mathcal{S}_k$,
3. $\left\{\sqrt{n}\right\}_{n=1}^\infty \notin \mathcal{S}_k$.

Spaces $H_k(\{e_n\})$, $k \in \mathbb{N}$

Choose separable Hilbert space H and let $\{e_n\}_1^\infty$ be an arbitrary Riesz basis in H. Then we define a Hilbert space $H_k(\{e_n\})$, $k \in \mathbb{N}$, as

$$H_k(\{e_n\}) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n : \ \{c_n\}_1^\infty \in \ell_2(\Delta^k) \right\}, \ k \in \mathbb{N},$$

with $\left\| (\mathfrak{f}) \sum\limits_{n=1}^{\infty} c_n e_n \right\|_k = \left\| \sum\limits_{n=1}^{\infty} \left(\Delta^k c_n\right) e_n \right\| = \left\| \sum\limits_{n=1}^{\infty} \sum\limits_{j=0}^{k} (-1)^j C_k^j c_{n-j} e_n \right\|.$

Here $\ell_2(\Delta^k) = \{\alpha = \{\alpha_n\}_{n=1}^\infty : \ \Delta^k \alpha \in \ell_2\}$.

The space $\ell_2(\Delta)$ was first introduced and studied by F. Başar & B. Altay, Ukrainian Math. J., 2003. Later, in 2006, the space $\ell_2(\Delta^k)$, $k \in \mathbb{N}$, was studied by B. Altay, Studia Sci. Math. Hungar.

$H_k(\{e_n\})$, $k \in \mathbb{N}$, is isomorphic to ℓ_2 and the following holds:

$H \subset H_1(\{e_n\}) \subset H_2(\{e_n\}) \subset H_3(\{e_n\}) \subset \dots$

Proposition (Sklyar & Marchenko)

1. $\overline{Lin}\{e_n\}_{n=1}^{\infty} = H_k(\{e_n\})$;
2. $\{e_n\}_{n=1}^{\infty}$ does not form a basis of $H_k(\{e_n\})$;
3. $\{e_n\}_{n=1}^{\infty}$ has a unique biorthogonal system
$$\left\{\chi_n = (I-T)^{-k}(I-T^*)^{-k} e_n^*\right\}_{n=1}^{\infty}$$
in $H_k(\{e_n\})$, where $Te_n = e_{n+1}$, $n \in \mathbb{N}$, and $\langle e_n, e_m^* \rangle = \delta_n^m$;
4. $\{\chi_n\}_{n=1}^{\infty}$ is uniformly minimal sequence in $H_k(\{e_n\})$, $\{e_n\}_{n=1}^{\infty}$ is minimal but not uniformly minimal in $H_k(\{e_n\})$;
5. $H_k(\{e_n\})$ is Hilbert space, isomorphic to ℓ_2;
6. $L = \left\{x = (\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n \in H_k(\{e_n\}) : \{c_n\}_{n=1}^{\infty} \in \ell_2(\Delta^k) \cap c_0\right\}$, is not a (closed) subspace of $H_k(\{e_n\})$.

Theorem (G.M. Sklyar & V. Marchenko, J. Funct. Anal., 2017)

The operator $A_k : H_k(\{e_n\}) \supset D(A_k) \mapsto H_k(\{e_n\}), k \in \mathbb{N}$, defined by

$$A_k x = A_k(\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} if(n) \cdot c_n e_n,$$

where $\{f(n)\}_{n=1}^{\infty} \in \mathcal{S}_k = \Big\{ \{f(n)\}_1^{\infty} : \lim_{n\to\infty} f(n) = +\infty;$ $\{n^j \Delta^j f(n)\}_{n=1}^{\infty} \in \ell_\infty \text{ for } 1 \le j \le k \Big\}$, with domain

$$D(A_k) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in H_k(\{e_n\}) : \{f(n) \cdot c_n\}_{n=1}^{\infty} \in \ell_2(\Delta^k) \right\},$$

generates the C_0-group $\{e^{A_k t}\}_{t\in\mathbb{R}}$ on $H_k(\{e_n\})$, which is given by

$$e^{A_k t} x = e^{A_k t}(\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} e^{itf(n)} c_n e_n, \; t \in \mathbb{R}. \qquad (3)$$

Multiple application of the discrete Hardy inequality for $p = 2$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2 \leq 4\sum_{n=1}^{\infty} a_n^2$$

plays the key role in the proof of this theorem.

In the proof we also use the Leibnitz theorem for finite differences,

$$\Delta^k(u_n v_n) = \sum_{j=0}^{k} C_k^j \Delta^{k-j} u_{n-j} \Delta^j v_n, \quad k \in \mathbb{N},$$

and the following formula,

$$\Delta^d c_n = \sum_{m=1}^{n} \Delta^{d+1} c_m, \quad d, n \in \mathbb{N}.$$

Proposition (Sklyar & Marchenko)

- The spectrum of A_k is $\sigma(A_k) = \sigma_p(A_k) = \{if(n)\}_1^\infty = \{\lambda_n\}_1^\infty \subset i\mathbb{R}$, it satisfies

$$\lim_{n\to\infty} i\lambda_n = -\infty, \qquad \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0,$$

  and the corresp. eigenvectors $\{e_n\}_{n=1}^\infty$ are dense and minimal, hence $\overline{D(A_k)} = H_k(\{e_n\})$, but do not form a Schauder basis.

- The resolvent of A_k is given by $(A_k - \lambda I)^{-1} x = (\mathfrak{f}) \sum_{n=1}^{\infty} \frac{c_n e_n}{if(n)-\lambda}$, $\lambda \in \rho(A_k) = \mathbb{C} \setminus \{if(n)\}_1^\infty$, where $x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in H_k(\{e_n\})$.

Remark

The sequence $\{f(n)\}_1^\infty$, although satisfies $\lim_{n\to\infty} f(n) = +\infty$, need not to be monotone and the spectrum $\sigma(A_k) = \sigma_p(A_k) = \{if(n)\}_1^\infty$ of A_k need not to be simple.

Proposition (Sklyar & Marchenko)

Let $k \in \mathbb{N}$ and $\left\{e^{A_k t}\right\}_{t\in\mathbb{R}}$ is the C_0-group from the above theorem. Then:

1. $\left\|e^{A_k t}\right\| \to \infty$, when $t \to \pm\infty$.
2. There exists a polynomial $\mathfrak{p}_k$ with positive coefficients, $\deg \mathfrak{p}_k = k$, such that for every $t \in \mathbb{R}$ we have

$$\left\|e^{A_k t}\right\| \leq \mathfrak{p}_k(|t|).$$

Also we study the questions posed at the beginning

in the Banach space setting and obtain similar answers!

To obtain these results we

- Introduce and study special classes of Banach spaces $\ell_{p,k}(\{e_n\})$, $p \geq 1$, $k \in \mathbb{N}$. Space $\ell_{p,k}(\{e_n\})$ depend on ℓ_p space and a chosen symmetric basis $\{e_n\}_1^\infty$ of ℓ_p.
- Prove that, if $p > 1$, then $\{e_n\}_1^\infty$ is dense and minimal in $\ell_{p,k}(\{e_n\})$ but not uniformly minimal, hence do not form a Schauder basis.
- Consider our classes of increasing sequences $\mathcal{S}_k$, $k \in \mathbb{N}$.

The concept of symmetric basis

was first introduced and studied by I. Singer, Revue de math. pures et appl., 1961, in connection with S. Banach's closed hyperplane problem and related question of C. Bessaga & A. Pelczynski from isomorphic theory of Banach spaces.

Spaces $\ell_{p,k}(\{e_n\})$, $p \geq 1$, $k \in \mathbb{N}$

Choose the space ℓ_p and let $\{e_n\}_1^\infty$ be an arbitrary symmetric basis in ℓ_p, $p \geq 1$. Then we define a Banach space $\ell_{p,k}(\{e_n\})$, $p \geq 1$, $k \in \mathbb{N}$, as

$$\ell_{p,k}(\{e_n\}) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n : \ \{c_n\}_{n=1}^{\infty} \in \ell_p(\Delta^k) \right\}, \ p \geq 1, \ k \in \mathbb{N},$$

with $\left\| (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \right\|_k = \left\| \sum_{n=1}^{\infty} (\Delta^k c_n)\, e_n \right\| = \left\| \sum_{n=1}^{\infty} \sum_{j=0}^{k} (-1)^j C_k^j c_{n-j} e_n \right\|.$

Here $\ell_p(\Delta^k) = \{\alpha = \{\alpha_n\}_{n=1}^\infty : \ \Delta^k \alpha \in \ell_p\}$.

$\ell_{p,k}(\{e_n\})$, $p \geq 1$, $k \in \mathbb{N}$, is isomorphic to ℓ_p and the following holds:

$\ell_p \subset \ell_{p,1}(\{e_n\}) \subset \ell_{p,2}(\{e_n\}) \subset \ell_{p,3}(\{e_n\}) \subset \dots$

Proposition (Sklyar & Marchenko)

1. If $p>1$, then $\overline{Lin}\{e_n\}_{n=1}^{\infty} = \ell_{p,k}(\{e_n\})$;
2. $\{e_n\}_{n=1}^{\infty}$ does not form a basis of $\ell_{p,k}(\{e_n\})$;
3. If $p>1$, then $\{e_n\}_{n=1}^{\infty}$ has a unique biorthogonal system
$$\left\{\chi_n = (I-T)^{-k}(I-T^*)^{-k} e_n^*\right\}_{n=1}^{\infty}$$
in $\left(\ell_{p,k}(\{e_n\})\right)^*$, where $Te_n = e_{n+1}$, $n\in\mathbb{N}$, and $\{e_n^*\}_{n=1}^{\infty}$ is biorthogonal to $\{e_n\}_{n=1}^{\infty}$ basis of ℓ_q, where $\frac{1}{p}+\frac{1}{q}=1$;
4. If $p>1$, then $\{\chi_n\}_{n=1}^{\infty}$ is uniformly minimal sequence in $\left(\ell_{p,k}(\{e_n\})\right)^*$ while the sequence $\{e_n\}_{n=1}^{\infty}$ is minimal but not uniformly minimal in $\ell_{p,k}(\{e_n\})$;
5. $L = \left\{x = (\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n \in \ell_{p,k}(\{e_n\}) : \{c_n\}_{n=1}^{\infty} \in \ell_p(\Delta^k)\cap c_0\right\}$ is not a (closed) subspace of $\ell_{p,k}(\{e_n\})$.

Theorem (G.M. Sklyar & V. Marchenko, J. Funct. Anal., 2017)

Let $\{e_n\}_{n=1}^{\infty}$ be a symmetric basis of ℓ_p, $p>1$ and $k\in\mathbb{N}$. Then $\{e_n\}_{n=1}^{\infty}$ does not form a Schauder basis of $\ell_{p,k}(\{e_n\})$ and the operator $A_k:\ell_{p,k}(\{e_n\})\supset D(A_k)\mapsto \ell_{p,k}(\{e_n\})$, defined by

$$A_k x = A_k(\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f})\sum_{n=1}^{\infty} if(n)\cdot c_n e_n,$$

where $\{f(n)\}_{n=1}^{\infty}\in\mathcal{S}_k$, with domain

$$D(A_k)=\left\{x=(\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n\in \ell_{p,k}(\{e_n\}):\ \{f(n)\cdot c_n\}_{n=1}^{\infty}\in \ell_p(\Delta^k)\right\},$$

generates the C_0-group $\{e^{A_k t}\}_{t\in\mathbb{R}}$ on $\ell_{p,k}(\{e_n\})$, which is given by

$$e^{A_k t}x = e^{A_k t}(\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f})\sum_{n=1}^{\infty} e^{itf(n)} c_n e_n. \qquad (4)$$

Multiple application of the discrete Hardy inequality for $p > 1$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p$$

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Proposition (Sklyar & Marchenko)

- The spectrum of A_k is $\sigma_p(A_k) = \{if(n)\}_1^\infty = \{\lambda_n\}_1^\infty \subset i\mathbb{R}$, it satisfies

$$\lim_{n\to\infty} i\lambda_n = -\infty, \qquad \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0,$$

  and the corresp. eigenvectors $\{e_n\}_{n=1}^\infty$ are dense and minimal, but do not form a Schauder basis.
- The resolvent of A_k is given by $(A_k - \lambda I)^{-1} x = (\mathfrak{f}) \sum_{n=1}^{\infty} \frac{c_n e_n}{if(n)-\lambda}$,

  $\lambda \in \rho(A_k) = \mathbb{C} \setminus \{if(n)\}_1^\infty$, where $x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in \ell_{p,k}(\{e_n\})$.

Proposition (Sklyar & Marchenko)

Let $\{\lambda_n\}_{n=1}^{\infty} \subset i\mathbb{R}$ satisfy

$$\lim_{n\to\infty} i\lambda_n = -\infty, \qquad \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0,$$

and $\exists \alpha \in \left(0, \frac{1}{2}\right]$:

$$\liminf_{n\to\infty} n^{\alpha} |\lambda_n - \lambda_{n-1}| > 0.$$

Then the operator A, defined by $Ax = A(\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f})\sum_{n=1}^{\infty} \lambda_n c_n e_n$, with domain $D(A) = \left\{ x = (\mathfrak{f})\sum_{n=1}^{\infty} c_n e_n \in H_l(\{e_n\}) : \{\lambda_n c_n\}_{n=1}^{\infty} \in \ell_2(\Delta) \right\}$, does not generate the C_0-semigroup on the space $H_l(\{e_n\})$.

Example

We can take $\lambda_n = i\sqrt{n}$, $n \in \mathbb{N}$.

Open questions:

- Is it possible to construct the unbounded generator of the C_0-group with purely imaginary eigenvalues not satisfying (2) and family of eigenvectors, which form a bounded non-Riesz basis in a Hilbert space?
- What natural evolution phenomena are described by a such kind of evolution equations?
- What happens between $i \ln n$ and $i\sqrt{n}$ in our constructions in $H_{\mathfrak{l}}(\{e_n\})$?
- How can the XYZ theorem be generalized to the case of some kinds of bases in Banach spaces, e.g. symmetric bases?

Thanks for the attention!

Thanks to the organizers of IWOTA-2017!