Toeplitz Operators on Polyanalytic Bergman Spaces Over the Upper-Half Plane

Hocine Guediri

King Saud University (Riyadh, KSA)

hguediri@ksu.edu.sa

August 18, 2017

Overview

Motivation

The class of Toeplitz operators represents a very interesting example of concrete operators.

On Hardy, Bergman, Dirichlet and Fock spaces (in $\mathbb{D}$ and $\mathbb{C}$) in one and several dimensions, this class of operators has been intensively studied from different points of view. The Harmonic case of these spaces and their operators has also been considered subsequently by many authors.

Recently Ž. Čučkovič and T. Le [Toeplitz operators on Bergman spaces of polyanalytic functions, Bull. London Math. Soc., 44(5), 961-973, 2012] have considered Toeplitz operators on polyanalytic Bergman spaces of the disk.

In this talk, we study compact Toeplitz operators on polyanalytic Bergman spaces over the complex upper half-plane.

Spaces

Let $\Pi := \{z = x + iy \in \mathbb{C},\ \text{Im } z = y > 0\}$ denote the upper complex half-space. For $n \geq 1$, a function f is polyanalytic of order n (n-analytic) on Π if it satisfies the generalized Cauchy-Riemann equation:

$$\frac{\partial^n f}{\partial \overline{z}^n} = 0 \text{ in } \Pi.$$

Spaces

Let $\Pi := \{z = x + iy \in \mathbb{C},\ \text{Im } z = y > 0\}$ denote the upper complex half-space. For $n \geq 1$, a function f is polyanalytic of order n (n-analytic) on Π if it satisfies the generalized Cauchy-Riemann equation:

$$\frac{\partial^n f}{\partial \overline{z}^n} = 0 \text{ in } \Pi.$$

The weighted Lebesgue space $L^2_\lambda(\Pi, d\mu_\lambda) = L^2_\lambda(\Pi)$ is the generic Lebesgue space of measurable functions that are square integrable over Π with respect to the measure

$$d\mu_\lambda = dx d\nu_\lambda(y) = (\lambda + 1)(2y)^\lambda dx dy,\ \lambda \in (-1, \infty)$$

with norm and inner product

$$||f||_2 = \left(\int_\Pi |f(z)|^2 d\mu_\lambda(z)\right)^{\frac{1}{2}} < \infty \quad ; \quad < f, g >= \int_\Pi f(z)\overline{g(z)} d\mu_\lambda(z)$$

Spaces

The polyanalytic Bergman space

The space

$$\mathcal{A}^2_{n,\lambda}(\Pi) = \left\{ f \in L^2_\lambda(\Pi),\ \frac{\partial^n f}{\partial \overline{z}^n} = 0 \ \text{ in } \Pi \ \right\}$$

is the closed subspace of $L^2_\lambda(\Pi)$ consisting of polyanalytic functions.

Spaces

The polyanalytic Bergman space

The space

$$\mathcal{A}^2_{n,\lambda}(\Pi) = \left\{ f \in L^2_\lambda(\Pi),\ \frac{\partial^n f}{\partial \overline{z}^n} = 0 \ \text{ in } \Pi \ \right\}$$

is the closed subspace of $L^2_\lambda(\Pi)$ consisting of polyanalytic functions.

It is a reproducing kernel Hilbert space, and its kernel function $K_{n,\lambda}(z,\zeta)$ has been established via Fourier transform by J.R. Ortega [Bol. Soc. Mat. Mexicana (3) 13, 2007]; it takes the form:

The reproducing kernel

$$K_{n,\lambda} = \frac{i(z-\overline{z}-\zeta+\overline{\zeta})^{-1}}{(i\overline{\zeta}-iz)^{\lambda+1}} \sum_{k=0}^{n} \sum_{j=0}^{n-1} \gamma_{j,k,n} \frac{(z-\overline{z})^j(\zeta-\overline{\zeta})^k-(z-\overline{z})^k(\zeta-\overline{\zeta})^j}{(z-\overline{\zeta})^{j+k}}$$

Spaces

with coefficients

$$\gamma_{j,k,n} = \frac{(-1)^{j+k}\Gamma(n+1)\Gamma(\lambda+n+1)\Gamma(\lambda+j+k+1)}{\pi(\lambda+1)\Gamma(\lambda+j+1)j!(n-1-j)!\Gamma(\lambda+k+1)k!(n-k)!}$$

Spaces

with coefficients

$$\gamma_{j,k,n} = \frac{(-1)^{j+k}\Gamma(n+1)\Gamma(\lambda+n+1)\Gamma(\lambda+j+k+1)}{\pi(\lambda+1)\Gamma(\lambda+j+1)j!(n-1-j)!\Gamma(\lambda+k+1)k!(n-k)!}$$

A symmetric form of the reproducing kernel $K_{n,\lambda}(z,\zeta)$ can also be given by

$$\frac{(z-\overline{z}-\zeta+\overline{\zeta})^{-1}}{(i\overline{\zeta}-iz)^{\lambda+2}}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\alpha_{j,k,n}\left(\frac{z-\overline{z}}{\lambda+j+1}-\frac{\zeta-\overline{\zeta}}{\lambda+k+1}\right)\frac{(z-\overline{z})^j(\zeta-\overline{\zeta})^k}{(z-\overline{\zeta})^{j+k}}$$

where $\alpha_{j,k,n} = (\lambda+n)\lambda_{j,k,n}$, with

$$\lambda_{j,k,n} = \frac{(-1)^{j+k}\Gamma(n)\Gamma(\lambda+n)\Gamma(\lambda+j+k+2)}{\pi(\lambda+1)\Gamma(\lambda+j+1)j!(n-1-j)!\Gamma(\lambda+k+1)k!(n-1-k)!}$$

Spaces

Let us calculate the norm of the reproducing kernel. For, observe that

$$K_{n,\lambda}(z,\zeta) = \sum_{q=1}^{n} K_{(q)\lambda}(z,\zeta)$$

with $K_{(q)\lambda}(z,\zeta)$ being the reproducing kernel of the true q-analytic function space

$$\mathcal{A}^2_{(q)\lambda}(\Pi) = \mathcal{A}^2_{q,\lambda} \ominus \mathcal{A}^2_{q-1,\lambda}(\Pi)$$

namely

$$K_{(q)\lambda}(z,\zeta) = \frac{1}{(i\overline{\zeta} - iz)^{\lambda+2}} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1} \lambda_{j,k,q} \left(\frac{z-\overline{z}}{z-\overline{\zeta}} \right)^{j} \left(\frac{\zeta-\overline{\zeta}}{z-\overline{\zeta}} \right)^{k}$$

On the other hand, we have

$$||K_{n,\lambda}(z)||^2 = K_{n,\lambda}(z,z) = \sum_{q=1}^{n} K_{(q)\lambda}(z,z).$$

But we know that

$$K_{(q)\lambda}(z,z) = \frac{1}{(2\ \mathrm{Im}(z))^{\lambda+2}} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1} \lambda_{j,k,q},$$

Thus, we see that

$$K_{n,\lambda}(z,z) = \frac{\lambda_n}{2^{\lambda+2}(\mathrm{Im}\ z)^{\lambda+2}},$$

with coefficients given by

$$\lambda_n = \sum_{q=1}^{n} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1} \lambda_{j,k,q} > 0$$

Hence, we obtain some facts about the polyanalytic normalized Bergman reproducing kernel:

Spaces

Facts about the normalized reproducing kernel

The normalized reproducing kernel is then given by

$$k_{n,\lambda}(z,\zeta) = \frac{K_{n,\lambda}(z,\zeta)}{||K_{n\lambda(z)}||}$$

where

$$||K_{n,\lambda}(z)|| \frac{\sqrt{\lambda_n}}{2^{1+\frac{\lambda}{2}}(\text{Im } z)^{1+\frac{\lambda}{2}}}, \text{(note that } ||K_{n\lambda}(z)|| \longrightarrow \infty \text{ as Im } z \longrightarrow 0).$$

The density of linear combinations of reproducing kernels (of the form $\sum_{j=0}^{m} \alpha_j K_{n,\lambda}(z_j)$) together with the latter yield the following asymptotic behavior of the normalized reproducing kernel

$$k_{n,\lambda}(z,.) \longrightarrow 0 \text{ weakly as } \text{Im } z \longrightarrow 0.$$

Toeplitz Operators

The Bergman projection is given by

$$\begin{aligned} \mathcal{P}: \ L^2_\lambda(\Pi) &\longrightarrow \mathcal{A}^2_{n,\lambda}(\Pi) \\ f &\longrightarrow \mathcal{P}f(z) = \int_\Pi f(\zeta)\overline{K}_{n,\lambda}(z,\zeta)d\mu_\lambda(\zeta). \end{aligned}$$

If f belongs to $L^2_\lambda(\Pi)$, then for any bounded $g \in \mathcal{A}^2_{n,\lambda}(\Pi)$, the pointwise product fg lies in $L^2_\lambda(\Pi)$.

Thus, the Toeplitz operator is defined as:

$$T_f g(z) = \mathcal{P}(fg)(z) = \int_\Pi f(w)g(w)\overline{K}_{n,\lambda}(z,w)d\mu_\lambda(w).$$

Toeplitz Operators

The Bergman projection is given by

$$\begin{array}{rcl} \mathcal{P}: \ L^2_\lambda(\Pi) & \longrightarrow & \mathcal{A}^2_{n,\lambda}(\Pi) \\ f & \longrightarrow & \mathcal{P}f(z) = \int\limits_{\Pi} f(\zeta)\overline{K}_{n,\lambda}(z,\zeta)d\mu_\lambda(\zeta). \end{array}$$

If f belongs to $L^2_\lambda(\Pi)$, then for any bounded $g \in \mathcal{A}^2_{n,\lambda}(\Pi)$, the pointwise product fg lies in $L^2_\lambda(\Pi)$.

Thus, the Toeplitz operator is defined as:

$$T_f g(z) = \mathcal{P}(fg)(z) = \int\limits_{\Pi} f(w)g(w)\overline{K}_{n,\lambda}(z,w)d\mu_\lambda(w).$$

Boundedness

If $f \in L^2_\lambda(\Pi) \cap L^\infty(\Pi)$, then T_f is a bounded operator on $\mathcal{A}^2_{n,\lambda}(\Pi)$ with $||T_f|| \leq ||f||_\infty$.

Toeplitz Operators

If $u \in L^2_\lambda(\Pi)$ is harmonic, then $u = f + \overline{g}$, with $f, g \in L^2_\lambda(\Pi)$ analytic functions. The cancellation condition is

$$\int\limits_{\Pi} f(w) d\mu_\lambda(w) = 0, \quad \text{for any } f \in L^1(\Pi, d\mu_\lambda) \text{ analytic.}$$

and thus, if $u = f + \overline{g}$ is harmonic, then for all $z, w \in \Pi$ we have

$$< T_u K_{n,\lambda}(z), K_{n,\lambda}(w) >= (f(w) + \overline{g}(z)) < K_{n,\lambda}(z), K_{n,\lambda}(w) > .$$

Thus, we get

$$< T_u K_{n,\lambda}(z), K_{n,\lambda}(z) >= u(z) ||K_{n,\lambda}(z)||^2, \; z \in \Pi.$$

So, if T_u is bounded, then

Toeplitz Operators

$$||u||_\infty \leq \sup_{z\in\Pi} |\,\langle T_u k_{n,\lambda}(z), k_{n,\lambda}(z)\rangle\,| \leq ||T_u||.$$

Thus, since $||T_u|| \leq ||u||_\infty$, we see that

Toeplitz Operators

$$||u||_\infty \leq \sup_{z\in\Pi} |\langle T_u k_{n,\lambda}(z), k_{n,\lambda}(z)\rangle| \leq ||T_u||.$$

Thus, since $||T_u|| \leq ||u||_\infty$, we see that

Norm of a class of Toeplitz operators

$||T_u|| = ||u||_\infty$ for bounded harmonic symbols.

Toeplitz Operators

$$||u||_\infty \leq \sup_{z\in\Pi} |\langle T_u k_{n,\lambda}(z), k_{n,\lambda}(z)\rangle| \leq ||T_u||.$$

Thus, since $||T_u|| \leq ||u||_\infty$, we see that

Norm of a class of Toeplitz operators

$||T_u|| = ||u||_\infty$ for bounded harmonic symbols.

Next, in a standard way, we see that:

Toeplitz Operators

$$||u||_\infty \leq \sup_{z\in\Pi} |\langle T_u k_{n,\lambda}(z), k_{n,\lambda}(z)\rangle| \leq ||T_u||.$$

Thus, since $||T_u|| \leq ||u||_\infty$, we see that

Norm of a class of Toeplitz operators

$$||T_u|| = ||u||_\infty \text{ for bounded harmonic symbols.}$$

Next, in a standard way, we see that:

Self-adjointness

The adjoint of T_f is given by $T_f^* = T_{\overline{f}}$, and so T_f is self-adjoint if and only if f is real-valued.

Toeplitz Operators

$$||u||_\infty \leq \sup_{z\in\Pi} |\langle T_u k_{n,\lambda}(z), k_{n,\lambda}(z)\rangle| \leq ||T_u||.$$

Thus, since $||T_u|| \leq ||u||_\infty$, we see that

Norm of a class of Toeplitz operators

$||T_u|| = ||u||_\infty$ for bounded harmonic symbols.

Next, in a standard way, we see that:

Self-adjointness

The adjoint of T_f is given by $T_f^* = T_{\overline{f}}$, and so T_f is self-adjoint if and only if f is real-valued.

Question about normality

Is it true that T_f is normal if and only if the range of f lies on a line in $\mathbb{C}$? (at least for bounded harmonic symbols).

Berezin transform

The Berezin symbol of an operator $T \in \mathscr{B}(\mathcal{A}_{n,\lambda}(\Pi))$ is defined to be the real analytic function on Π given by

$$\mathcal{B}(T)(z) := \langle Tk_{n,\lambda}(z), k_{n,\lambda}(z) \rangle ,$$

and the underlying map is the Berezin transform of T.

The Berezin symbol of an operator $T \in \mathscr{B}(\mathcal{A}_{n,\lambda}(\Pi))$ is defined to be the real analytic function on Π given by

$$\mathcal{B}(T)(z) := \langle Tk_{n,\lambda}(z), k_{n,\lambda}(z)\rangle ,$$

and the underlying map is the Berezin transform of T.
So, for a Toeplitz operator, we have

$$\begin{aligned}\mathcal{B}(T_f)(z) &= \int_{\Pi} f(\zeta)|k_{n,\lambda}(z,\zeta)|^2 d\mu_\lambda(\zeta) \\ &= \frac{(2\text{Im } z)^{\lambda+2}}{\lambda_n}\int_{\Pi} f(\zeta)\frac{|z-\overline{z}+\zeta-\overline{\zeta}|^{-2}}{|\overline{\zeta}-z|^{2\lambda+4}}\Lambda(z,\zeta)d\mu_\lambda(\zeta).\end{aligned}$$

where $\Lambda(z,\zeta)$ is given by

$$\Lambda(z,\zeta) = \left|\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\alpha_{j,k,n}\left(\frac{z-\overline{z}}{\lambda+j+1}-\frac{\zeta-\overline{\zeta}}{\lambda+k+1}\right)\frac{(z-\overline{z})^j(\zeta-\overline{\zeta})^k}{(z-\overline{\zeta})^{j+k}}\right|^2$$

Pseudo-hyperbolic disks

Let $w = s + it \in \Pi$, and define $\varphi_w : \ \Pi \longrightarrow \Pi$ by

$$\varphi_w(z) = \frac{x - s}{t} + i\frac{y}{t}, \quad z = x + iy.$$

Then φ_w is one to one, onto and holomorphic on Π.
And, define the pseudo-hyperbolic distance $\rho(w, z)$ by

$$\rho(w, z) = \left| \frac{z - w}{z - \overline{w}} \right|, \ \text{ on } \ \Pi,$$

which is a metric on Π. Let $B(z, r)$ be the Euclidean disk, and for $w = s + it \in \Pi$ and $0 < R < 1$, let

$$\Delta(w, R) := \{z \in \mathbb{C}, \ \rho(z, w) < R\}$$

be the pseudo-hyperbolic disk with center w and radius R. Then, we infer that

$$z \in \Delta(w, R) \iff \rho(z, w) < R \iff z \in B\left(\left(s, \frac{1 + R^2}{1 - R^2}t\right), \frac{2Rt}{1 - R^2}\right).$$

Pseudo-hyperbolic disks

Thus, we have the following:

Proposition 1

Let $w = s + it \in \Pi$ and $0 < R < 1$. Since the pseudo-hyperbolic disk $\Delta(w, R)$ coincides with the Euclidean disk $B\left(\left(s, \frac{1+R^2}{1-R^2}t\right), \frac{2Rt}{1-R^2}\right)$, we see that

$$|\Delta(w, R)| = \frac{4\pi R^2 t^2}{(1-R^2)^2}.$$

and $\forall z \in \Delta(w, R)$, we have

$$\frac{1-R}{1+R} < \frac{\text{Im } w}{\text{Im } z} < \frac{1+R}{1-R}.$$

Compact Toeplitz Operators

Bounded compactly supported symbols yield compact Toeplitz operators:

Compact Toeplitz Operators

Bounded compactly supported symbols yield compact Toeplitz operators:

Lemma 2

Let $\varphi \in L^\infty(\Pi)$ such that $\mathrm{Supp}(\varphi)$ is a compact subset of Π. Then, the compression of the multiplication operator $\mathcal{M}_\varphi \big/ \mathcal{A}^2_{n,\lambda}(\Pi)$ is compact.

Compact Toeplitz Operators

Bounded compactly supported symbols yield compact Toeplitz operators:

Lemma 2

Let $\varphi \in L^{\infty}(\Pi)$ such that $\mathrm{Supp}(\varphi)$ is a compact subset of Π. Then, the compression of the multiplication operator $\mathcal{M}_{\varphi} \diagup_{\mathcal{A}^2_{n,\lambda}(\Pi)}$ is compact.

Since $T_{\varphi} = \mathcal{P}\mathcal{M}_{\varphi}$, the latter yields:

Compact Toeplitz Operators

Bounded compactly supported symbols yield compact Toeplitz operators:

Lemma 2

Let $\varphi \in L^\infty(\Pi)$ such that $\mathrm{Supp}(\varphi)$ is a compact subset of Π. Then, the compression of the multiplication operator $\mathcal{M}_\varphi \diagup_{\mathcal{A}^2_{n,\lambda}(\Pi)}$ is compact.

Since $T_\varphi = \mathcal{P}\mathcal{M}_\varphi$, the latter yields:

Proposition 3

If φ is a compactly supported bounded function on Π, then the Toeplitz operator T_φ is compact on $\mathcal{A}^2_{n,\lambda}(\Pi)$.

Compact Toeplitz Operators

Bounded compactly supported symbols yield compact Toeplitz operators:

Lemma 2

Let $\varphi \in L^\infty(\Pi)$ such that $\mathrm{Supp}(\varphi)$ is a compact subset of Π. Then, the compression of the multiplication operator $\mathcal{M}_\varphi \big/ {\mathcal{A}^2_{n,\lambda}(\Pi)}$ is compact.

Since $T_\varphi = \mathcal{P}\mathcal{M}_\varphi$, the latter yields:

Proposition 3

If φ is a compactly supported bounded function on Π, then the Toeplitz operator T_φ is compact on $\mathcal{A}^2_{n,\lambda}(\Pi)$.

Now, let $\mathscr{C}_0(\Pi)$ denote the algebra of continuous functions on Π such that $f(z) \longrightarrow 0$ as $\mathrm{Im}\, z \longrightarrow 0$.
For Toeplitz operators with such symbols, we have:

Compact Toeplitz Operators

Corollary 4

Let φ be in $\mathscr{C}_0(\Pi)$. Then, T_φ is compact on $\mathcal{A}^2_{n,\lambda}(\Pi)$.

Compact Toeplitz Operators

Corollary 4

Let φ be in $\mathscr{C}_0(\Pi)$. Then, T_φ is compact on $\mathcal{A}^2_{n,\lambda}(\Pi)$.

The reproducing kernel satisfies the following estimate:

Compact Toeplitz Operators

Corollary 4

Let φ be in $\mathscr{C}_0(\Pi)$. Then, T_φ is compact on $\mathcal{A}^2_{n,\lambda}(\Pi)$.

The reproducing kernel satisfies the following estimate:

Lemma 5

For $\lambda > -1$ and $0 < \delta < 1$, there is $C_\delta > 0$ such that $|K_{n,\lambda}(z,\zeta)| \leq C_\delta, \ \forall z, \zeta \in \Pi, \text{ with } |z - \zeta| \geq \delta.$

Compact Toeplitz Operators

Corollary 4

Let φ be in $\mathscr{C}_0(\Pi)$. Then, T_φ is compact on $\mathcal{A}^2_{n,\lambda}(\Pi)$.

The reproducing kernel satisfies the following estimate:

Lemma 5

For $\lambda > -1$ and $0 < \delta < 1$, there is $C_\delta > 0$ such that $|K_{n,\lambda}(z,\zeta)| \leq C_\delta, \; \forall z, \zeta \in \Pi, \;$ with $|z - \zeta| \geq \delta.$

For $\mathscr{C}(\overline{\Pi})$ symbols, using Lemma 5, the sufficient condition of Corollary 4 turns out to be also necessary:

Compact Toeplitz Operators

Corollary 4

Let φ be in $\mathscr{C}_0(\Pi)$. Then, T_φ is compact on $\mathcal{A}^2_{n,\lambda}(\Pi)$.

The reproducing kernel satisfies the following estimate:

Lemma 5

For $\lambda > -1$ and $0 < \delta < 1$, there is $C_\delta > 0$ such that $|K_{n,\lambda}(z,\zeta)| \leq C_\delta, \ \forall z, \zeta \in \Pi, \$ with $|z - \zeta| \geq \delta$.

For $\mathscr{C}(\overline{\Pi})$ symbols, using Lemma 5, the sufficient condition of Corollary 4 turns out to be also necessary:

Proposition 6

Let $\varphi \in \mathscr{C}(\overline{\Pi})$. Then, the Toeplitz operator T_φ is compact on $\mathcal{A}^2_{n,\lambda}(\Pi)$ if and only if $\varphi \in \mathscr{C}_0(\Pi)$.

Compact Toeplitz Operators

The following subharmonicity behavior of polyanalytic functions, which is due to [Borichev & Hedenmalm: Adv. Math., 264 (2014), 464–505], is needed:

Compact Toeplitz Operators

The following subharmonicity behavior of polyanalytic functions, which is due to [Borichev & Hedenmalm: Adv. Math., 264 (2014), 464–505], is needed:

Lemma 7

Let $0 < p < \infty$ and suppose that u is N-harmonic in the disk $\mathcal{D}(z_0, r)$, where $z_0 \in \mathbb{C}$ and the radius r is positive. Then, there exist positive constants $C_1(N, P)$ and $C_2(N, P)$ depending only on N and p such that

$$|u(z)|^p \leq \frac{C_1(N,p)}{r^2} \int\limits_{\mathcal{D}(z_0,r)} |u(z)|^p dA(z), \quad z \in \mathcal{D}\left(z_0, \frac{r}{2}\right).$$

In particular, the inequality is satisfied at $z = z_0$, and we have

$$|\nabla u(z)|^p \leq \frac{C_2(N,p)}{r^{2+p}} \int\limits_{\mathcal{D}(z_0,r)} |u(z)|^p dA(z), \quad z \in \mathcal{D}\left(z_0, \frac{r}{2}\right).$$

Compact Toeplitz Operators

Next, the beautiful identity

$$1 - \rho^2(z, \zeta) = \frac{4(\text{Im } z)(\text{Im } \zeta)}{|z - \overline{\zeta}|^2}$$

is used to show that the normalized reproducing kernel has a nice behavior in the pseudo-hyperbolic framework:

Next, the beautiful identity

$$1-\rho^2(z,\zeta)=\frac{4(\text{Im } z)(\text{Im } \zeta)}{|z-\overline{\zeta}|^2}$$

is used to show that the normalized reproducing kernel has a nice behavior in the pseudo-hyperbolic framework:

Lemma 8

Let $z \in \Pi$, then there exists $0 < s < 1$ such that

$$|k_{n,\lambda}(z,\zeta)| \approx (\text{Im } \zeta)^{-1-\frac{\lambda}{2}}, \quad \text{for} \quad \zeta \in \Delta(z,s).$$

Compact Toeplitz Operators

Next, the beautiful identity

$$1 - \rho^2(z,\zeta) = \frac{4(\text{Im } z)(\text{Im } \zeta)}{|z-\overline{\zeta}|^2}$$

is used to show that the normalized reproducing kernel has a nice behavior in the pseudo-hyperbolic framework:

Lemma 8

Let $z \in \Pi$, then there exists $0 < s < 1$ such that

$$|k_{n,\lambda}(z,\zeta)| \approx (\text{Im } \zeta)^{-1-\frac{\lambda}{2}}, \quad \text{for} \quad \zeta \in \Delta(z,s).$$

Combining all this stuff, we arrive to the following characterization of compact Toeplitz operators with nonnegative symbols:

Compact Toeplitz Operators

Theorem 9

Let f be a nonnegative function from $L^1(\Pi, d\mu_\lambda)$. Then, the Toeplitz operator $T_f : \mathcal{A}^2_{n,\lambda}(\Pi) \longrightarrow \mathcal{A}^2_{n,\lambda}(\Pi)$ is compact if and only if the berezin symbol $\widetilde{f}(z) := \mathcal{B}(f) := \mathcal{B}(T_f) \longrightarrow 0$ as $\operatorname{Im} z \longrightarrow 0$.

Compact Toeplitz Operators

Theorem 9

Let f be a nonnegative function from $L^1(\Pi, d\mu_\lambda)$. Then, the Toeplitz operator $T_f : \mathcal{A}^2_{n,\lambda}(\Pi) \longrightarrow \mathcal{A}^2_{n,\lambda}(\Pi)$ is compact if and only if the berezin symbol $\widetilde{f}(z) := \mathcal{B}(f) := \mathcal{B}(T_f) \longrightarrow 0$ as $\text{Im } z \longrightarrow 0$.

Sketch of the proof:

The only if part is clear from the weak convergence of the normalized reproducing kernel established in Section 2.

Theorem 9

Let f be a nonnegative function from $L^1(\Pi, d\mu_\lambda)$. Then, the Toeplitz operator $T_f : \mathcal{A}^2_{n,\lambda}(\Pi) \longrightarrow \mathcal{A}^2_{n,\lambda}(\Pi)$ is compact if and only if the berezin symbol $\widetilde{f}(z) := \mathcal{B}(f) := \mathcal{B}(T_f) \longrightarrow 0$ as $\operatorname{Im} z \longrightarrow 0$.

Sketch of the proof:

The only if part is clear from the weak convergence of the normalized reproducing kernel established in Section 2.

For the if part, consider the Toeplitz operator with symbol $\widetilde{f}$. By Fubini's theorem and Lemma 8, we have

$$\langle T_{\widetilde{f}} g, g \rangle \geq \int\limits_{\Pi} f(w) \left(\frac{c}{(\operatorname{Im} w)^{2+\lambda}} \int\limits_{\Delta(w,r)} |g(z)|^2 (1+\lambda)(2\operatorname{Im} z)^\lambda dA(z) \right) d\mu_\lambda(w)$$

Making use of the subharmonic behavior of Lemma 7, some pseudohyperbolic disk estimates as well as Proposition 1, we infer that the inside integral of the latter is bigger than

$$c(r, \lambda, n)|g(w)|^2$$

Making use of the subharmonic behavior of Lemma 7, some pseudohyperbolic disk estimates as well as Proposition 1, we infer that the inside integral of the latter is bigger than

$$c(r,\lambda,n)|g(w)|^2$$

Thus, some more calculations lead us to the estimate

$$\left\langle T_{\widetilde{f}}g,g\right\rangle \geq c\left\langle T_f g,g\right\rangle, \quad \forall g \in \mathcal{A}^2_{n,\lambda}(\Pi).$$

Compact Toeplitz Operators

Making use of the subharmonic behavior of Lemma 7, some pseudohyperbolic disk estimates as well as Proposition 1, we infer that the inside integral of the latter is bigger than

$$c(r, \lambda, n)|g(w)|^2$$

Thus, some more calculations lead us to the estimate

$$\left\langle T_{\widetilde{f}} g, g \right\rangle \geq c \left\langle T_f g, g \right\rangle, \quad \forall g \in \mathcal{A}^2_{n,\lambda}(\Pi).$$

Now, if $\widetilde{f}(z) \longrightarrow 0$ as $\operatorname{Im} z \longrightarrow 0$, then $T_{\widetilde{f}}$ is a Toeplitz operator whose symbol is in $\mathscr{C}_0(\Pi)$, whence compact by Corollary 4. Thus, if $\{g_m\}$ is weakly convergent in $\mathcal{A}^2_{n,\lambda}(\Pi)$, then we see from the above inequality that $\langle T_f g_m, g_m \rangle \to 0$ as $m \to \infty$; whence T_f is compact on $\mathcal{A}^2_{n,\lambda}(\Pi)$.

Compact Toeplitz Operators

Theorem 10

Let f be a nonnegative function from $L^2(\Pi, d\mu_\lambda)$. Then, the Toeplitz operator T_f is compact on $\mathcal{A}^2_{n,\lambda}(\Pi)$ if and only if the compression $\mathcal{M}_f : \mathcal{A}^2_{n,\lambda}(\Pi) \longrightarrow L^2(\Pi, d\mu_\lambda)$ is compact.

Compact Toeplitz Operators

Theorem 10

Let f be a nonnegative function from $L^2(\Pi, d\mu_\lambda)$. Then, the Toeplitz operator T_f is compact on $\mathcal{A}^2_{n,\lambda}(\Pi)$ if and only if the compression $\mathcal{M}_f : \mathcal{A}^2_{n,\lambda}(\Pi) \longrightarrow L^2(\Pi, d\mu_\lambda)$ is compact.

Sketch of the proof:

Since $T_\varphi = \mathcal{P}\mathcal{M}_\varphi$, the if part is clear.

For the only if part, consider a sequence $\{g_m\}$ weakly convergent to 0 in $\mathcal{A}^2_{n,\lambda}(\Pi)$, and observe that

$$\langle T_f g_m, g_m \rangle = \langle f g_m, g_m \rangle = \left\langle \sqrt{f} g_m, \sqrt{f} g_m \right\rangle = ||\mathcal{M}_{\sqrt{f}} g_m||_2^2$$

where $\sqrt{f}$ is the positive square root of f.

Since $T_f g_m \to 0$ strongly, we get $\lim\limits_{m \to \infty} ||\mathcal{M}_{\sqrt{f}} g_m||_2 = 0$; whence $\mathcal{M}_{\sqrt{f}} : \ \mathcal{A}^2_{n,\lambda}(\Pi) \longrightarrow L^2(\Pi, d\mu_\lambda)$ is compact. It follows that $T_f = \mathcal{P}\mathcal{M}_{\sqrt{f}}\mathcal{M}_{\sqrt{f}}$ is compact on $\mathcal{A}^2_{n,\lambda}(\Pi)$.

Thank you for your kind attention