The logarithmic residue theorem

in higher dimensions:

following an early lead by Marinus A. Kaashoek

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Joint work with

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INTRODUCTION

What is the Logarithmic Residue Theorem?

Concerned with

$$\text{LogRes}(f;\Gamma) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f'(\lambda)}{f(\lambda)}\, d\lambda$$

f scalar analytic on open neighborhood of closure of inner domain of positively oriented closed contour Γ, on which f has no zeros (**integral well-defined**)

Fact:
$\text{LogRes}(f;\Gamma)$ equal to number of zeros of f inside Γ (multiplicities counted)

Observation 1

LogRes$(f;\Gamma)$ nonnegative integer

Observation 2

LogRes$(f;\Gamma) = 0 \quad \Rightarrow \quad f$ **nonzero** on inner domain Γ

Alternative formulation:

$$\text{LogRes}(f;\Gamma) = 0 \Rightarrow f \text{ **invertible values** on inner domain } \Gamma$$

ISSUE

What can be said for analytic functions f having their values in a complex Banach algebra (with unit element)**?**

Generally lack of commutativity

Two possibilities:

Left version: $$\frac{1}{2\pi i}\int_\Gamma f'(\lambda)f(\lambda)^{-1}\,d\lambda$$

Right version: $$\frac{1}{2\pi i}\int_\Gamma f(\lambda)^{-1}f'(\lambda)\,d\lambda$$

Focus on **left** version

From now on:

$\mathcal{B}$ Banach algebra (with unit element)

$$\mathrm{LogRes}(f;\mathcal{B};\Gamma) = \frac{1}{2\pi i}\int_\Gamma f'(\lambda)f(\lambda)^{-1}\,d\lambda$$

f takes invertible values on Γ

Terminology: **logarithmic residue** of f with respect to Γ

Issues suggested by the two earlier observations:

ISSUE 1

What kind of elements are the logarithmic residues in $\mathcal{B}$?

ISSUE 2

What can be said when $\mathrm{LogRes}(f;\mathcal{B};\Gamma)$ vanishes?

Surprisingly many ramifications

Focus of this talk

HISTORICAL BACKGROUND

1966: Lothrop Mittenthal
Operator-Valued Polynomials in a Complex Variable, and Generalizations of Spectral Theory, Thesis
(commutative case)

1969: **Suggestion Marinus A. Kaashoek**
PhD project H. Bart

1974: H. Bart
Spectral properties of locally holomorphic vector-valued functions
Correction imperfection in Mittenthal

Not directly addressing the two issues but with some relevance for them:

1970: A.S. Markus and E.I. Sigal

1971: I.C. Gohberg and E.I. Sigal

1978: H. Bart, D.C. Lay, M.A. Kaashoek

Restart around 1990:
together with Torsten Ehrhardt and Bernd Silbermann

HINTS

Spectral case:

$$f(\lambda) = \lambda e_{\mathcal{B}} - t, \qquad f'(\lambda) = e_{\mathcal{B}}$$

$$\mathsf{LogRes}(f; \mathcal{B}; \Gamma) = \frac{1}{2\pi i} \int_{\Gamma} f'(\lambda) f(\lambda)^{-1} \, d\lambda$$

$$= \frac{1}{2\pi i} \int_{\Gamma} (\lambda e_{\mathcal{B}} - t)^{-1} \, d\lambda$$

Spectral idempotent of t with respect to Γ

Vanishes $\Leftrightarrow$ t has no spectrum inside Γ

Generalization to 'pencil':

$$f(\lambda) = \lambda s - t, \qquad f'(\lambda) = s$$

Possibly $st \neq ts$**; no invertibility condition on** s

$$\mathsf{LogRes}(f;\mathcal{B};\Gamma) = \frac{1}{2\pi i}\int_\Gamma f'(\lambda)f(\lambda)^{-1}\,d\lambda$$
$$= \frac{1}{2\pi i}\int_\Gamma s(\lambda s - t)^{-1}\,d\lambda$$

Again **idempotent** (Stummel, **1974**)

Vanishes $\Leftrightarrow$ $\lambda s - t$ **invertible** λ inside Γ

ISSUE 1

What kind of elements are the logarithmic residues in $\mathcal{B}$?

Relationship with idempotents?

ISSUE 2

What can be said when $\mathrm{LogRes}(f;\mathcal{B};\Gamma)$ vanishes?

Invertibility $f(\lambda)$ for λ inside Γ?

LOGARITHMIC RESIDUES

and

SUMS OF IDEMPOTENTS

General observation (simple):

Each sum of idempotents in a Banach algebra $\mathcal{B}$ is a logarithmic residue in $\mathcal{B}$.

QUESTION

Are logarithmic residues always sums of idempotents?

Often they are, but not always

There is a simple counterexample (subalgebra of $\mathbb{C}^{3\times 3}$)

Often they are . . .

Spectral case and 'Stummel' (already mentioned)

- The commutative case
- The full matrix algebra $\mathcal{B} = \mathbb{C}^{n \times n}$
- Many **zero pattern** subalgebras of $\mathbb{C}^{n \times n}$

$\mathcal{B}$ **commutative**

Reduction analytic function to polynomial

Uses (nonelementary) **Gelfand Theory**

Multiplicative linear functionals $\mu : \mathcal{B} \to \mathbb{C}$

Also essential role for the famous **Newton's identities** for symmetric polynomials

The full matrix algebra $\mathcal{B} = \mathbb{C}^{n\times n}$

Function f with values in $\mathbb{C}^{n\times n}$

Key observation: rank/trace condition satisfied

$$\text{rank}\left(\text{LogRes}(f;\mathcal{B};\Gamma)\right) \leq \text{trace}\left(\text{LogRes}(f;\mathcal{B};\Gamma)\right) \in \mathbb{Z}_+$$

1990 Characterization by Hartwig/Putcha, independently by Wu

$$\Downarrow$$

LogRes$(f;\mathcal{B};\Gamma)$ is sum of (rank one) idempotents

The logarithmic residues in $\mathbb{C}^{n\times n}$ are precisely the sums of (rank one) idempotents in $\mathbb{C}^{n\times n}$

Subalgebras of $\mathbb{C}^{n\times n}$ determined by a pattern of zeros

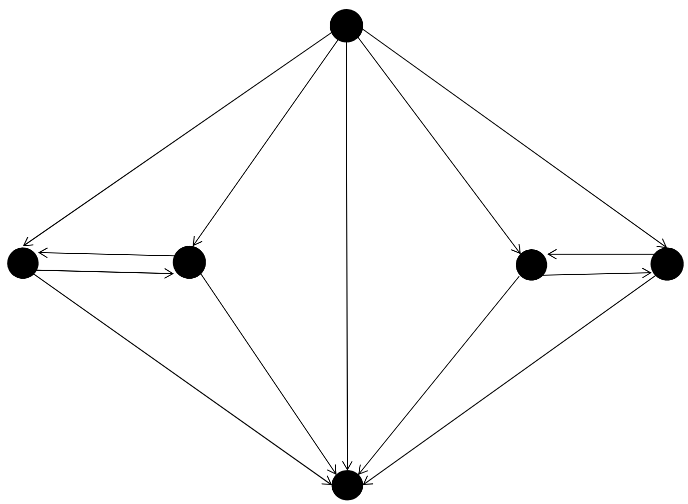
Typical example: matrices of the type

$$\begin{pmatrix} * & * & * & * & * & * \\ 0 & * & * & 0 & 0 & * \\ 0 & * & * & 0 & 0 & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

Stars: possibly nonzero

Pattern: preorder (reflexive / transitive)

Corresponding graph:



Enters graph theory

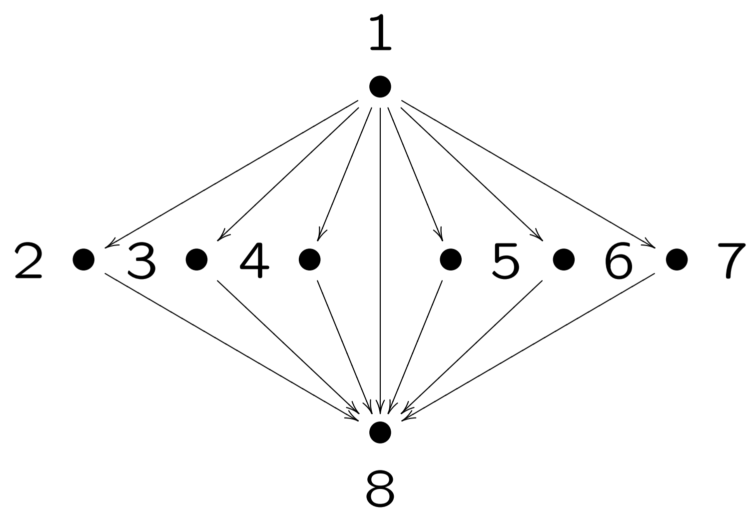
Many positive results
logarithmic residue $\Leftrightarrow$ **sum of idempotents**

Especially for patterns determined by a **partial order**
(reflexive, transitive, **antisymmetric**)

Example:

$$\begin{pmatrix} * & * & * & * & * & * & * & * \\ 0 & * & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

Determining graph:



Key case (underlying most of the positive results):

algebra of **block upper triangular** matrices
(fixed block size)

Typical example: block sizes 3,2,1 and 2

$$\begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{pmatrix}$$

Special case: **upper triangular** matrices (block sizes all 1)

Reduces to proving existence **nonnegative** **integer** solution of set of linear equations

Integer Programming

Works because of **total unimodularity**
Sufficient to establish existence of **nonnegative** **real** solution

Involves the famous **Farkas Lemma**

Acknowledgement: Albert Wagelmans (Rotterdam)

SPECTRAL REGULARITY

ISSUE 2

$$\mathrm{LogRes}(f; \mathcal{B}; \Gamma) = \frac{1}{2\pi i} \int_{\Gamma} f'(\lambda) f(\lambda)^{-1} d\lambda = 0$$

$\Downarrow$ **?**

$f(\lambda)$ invertible in $\mathcal{B}$ for all $\lambda \in$ inner domain Γ

Two levels:

individual functions

Banach algebras / **spectral regularity**

For certain Banach algebras true

For others not

Again many ramifications

Connections with different parts of mathematics.

NB **Trouble spot** in Thesis Mittenthal (**1966**)

Concerned with the **commutative case**

Correction H. Bart (**1974**)

Standard Gelfand Theory gives **spectral regularity**

Also **spectrally regular**:

Full matrix algebra $\mathbb{C}^{n\times n}$

Can be derived from **Markus/Sigal** (**1970**)

Generalizes to **Fredholm** valued **functions**

Can be derived from **Gohberg/Sigal** (**1971**)

Extends to all finite dimensional Banach algebras

Further with finite dimensional flavor:

Approximately finite-dimensional Banach algebras

dense union of finite dimensional subalgebras

Many interesting instances

Example:

The irrational rotation algebra

The Banach algebra generated by the compacts

$\mathcal{L}_{\mathcal{K}}(X) = \{\alpha I_X + K \mid \alpha \in \mathbb{C},\, K \in \mathcal{K}(X)\}$

$\mathcal{K}(X)$: compact operators on Banach space X

Note:

Spectacular result **Argyros/Haydon 2011**:
there is a Banach space Z for which $\mathcal{L}_{\mathcal{K}}(Z) = \mathcal{L}(Z)$

(each bounded linear operator on Z of the form $\alpha I_Z +$ compact)

$\Rightarrow$ $\mathcal{L}(Z)$ is spectrally regular

More about spectral regularity of $\mathcal{L}(X)$ later

More positive answers / Gelfand theory flavor:

Noncommutative Gelfand Theory

Multiplicative linear functionals $\longrightarrow$ matrix representations

Polynomial identity Banach algebras

Generalization of commutative Banach algebras

Uses **Krupnik** (**1987**)

Upper triangular operators on ℓ_2

KEY QUESTION:

What about $\mathcal{L}(\ell_2)$ itself?

Brings us to the last topic:

BANACH ALGEBRAS

FAILING TO BE

SPECTRALLY REGULAR

Indeed: $\mathcal{L}(\ell_2)$ is **not** spectrally regular

For a long time essentially the only example we had

Goes via construction of **nontrivial zero sum of idempotents**

Background observation:
In a spectrally regular Banach algebra zero sums of idempotents are trivial
(all summands zero)

Further investigation / recent years:
determining property: ℓ_2 isomorphic to ℓ_2^2

In fact:
if a Banach space X is isomorphic to X^{k+1} for some positive integer k,
then $\mathcal{L}(X)$ features a **non-trivial zero sum of idempotents** (projections),
hence the operator algebra $\mathcal{L}(X)$ is **not** spectrally regular

Yields lots of examples

In passing:
here arise issues of the type:
X **is** isomorphic to X^{k+1} but **not** to X^k

Work of **W.T Gowers**, winner Fields Medal (**1998**)

Striking illustration:

If S is an **uncountable**, **compact**, and **metrizable** topological space,
and $X = \mathcal{C}(S;\mathbb{C})$ is the the Banach space of continuous complex functions on S,
then the operator algebra $\mathcal{L}(X)$ is **not** spectrally regular

Indeed, $\mathcal{C}(S;\mathbb{C}) = \mathcal{C}(S;\mathbb{C})^2$

Corollary of a **truly remarkable** result from general topology by **A.A. Miljutin (1966)**:

THEOREM.

If S is an uncountable compact metrizable topological space and K is the usual Cantor set, then $\mathcal{C}(S;\mathbb{C})$ and $\mathcal{C}(K;\mathbb{C})$ are isomorphic **(Banach spaces)**.

Reformulation:

$\mathcal{C}(S;\mathbb{C})$ up to isomorphism independent of choice S

be it, for instance, K or $[0,1]$ **!**

Use now that $\mathcal{C}(K;\mathbb{C}) = \mathcal{C}(K;\mathbb{C})^2$

Obvious from the fact that the Cantor set is homeomorphic to the topological direct sum of 2 copies of itself

Thank you

for

your attention!

APPENDIX

1) Note about the two Issues

Many problems about analytic vector-valued functions can be reduced to the spectral pencil case $\lambda I - T$ via linearization by equivalence and extension (cf. pages 7 and 12).

Not so here!

The trouble comes from the multiplicativity aspect in forming the logarithmic derivative which features under the integral in the definition of the logarithmic residue (see page 6).

2)

Mentioned on page 26: **W.T. Gowers** gave examples of Banach spaces X for which X is isomorphic to X^{k+1} but not to X^k. Here k may be any integer larger than or equal to 2. These examples are complicated.

Using an **alternative approach to constructing Cantor sets**, it is possible to produce relatively simple examples of Banach spaces Y for which Y is isomorphic to Y^{k+1} but for which it is **not at all clear** whether or not Y is isomorphic to X^k.

See: H. Bart, T. Erhardt, B. Silbermann: *Zero sums of idempotents and Banach algebras failing to be spectrally regular*, In: Operator Theory: Advances and Applications, Vol. 237, Birkhäuser Verlag, Springer Basel AG (2013), 41-78.