Characterization of multi parameter BMO spaces through commutators

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August 2017

Hankel vs. Toeplitz on $\mathbb{T}$

$P_\pm$ projection operator onto non-negative and negative frequencies.
A Hankel operator with symbol b is

$$H_b : L^2_+ \to H^2_-, \ f \mapsto P_- b P_+ f$$

$b \in \mathrm{BMO}$ characterises boundedness.
A Toeplitz operator with symbol b is

$$T_b : L^2_+ \to H^2_+, \ f \mapsto P_+ b P_+ f$$

$b \in L^\infty$ characterises boundedness.

Hankel Operators $P_- b P_+$

$$\|H_b\|$$
$$= \sup_{\|g\|_{H^2_-}=1} \sup_{\|f\|_{H^2}=1} |(H_b f, g)|$$
$$= \sup_{\|g\|_{H^2_-}=1} \sup_{\|f\|_{H^2}=1} |(P_-(bf), g)|$$
$$= \sup_{\|g\|_{H^2_-}=1} \sup_{\|f\|_{H^2_+}=1} |(P_- bf, g)|$$
$$= \sup_{\|g\|_{H^2}=1} \sup_{\|f\|_{H^2}=1} |(P_- b, \bar{f} g)|$$

Anti-analytic part of b defines bounded linear functional on $H^1 \subset L^1$. Extend by Hahn Banach to all of L^1, i.e. a bounded function with the same anti-analytic part as b.

Using $H^1 - BMO$ duality, we get the characterisation of BMO.

Toeplitz operators $P_+ bP_+$

Clearly

$$||T_b|| \leq ||b||_\infty$$

But L^∞ also characterises boundedness:

It is easy to see that

$$\bar{\lambda}^n P_+ \lambda^n \to I$$

in L^2 in SOT. Nothing happens to such f with FS cut off at $-n$.
So

$$\bar{\lambda}^n P_+ bP_+ \lambda^n \to b$$

in L^2 in SOT.
Now as multiplication operators:

$$\|b\| \leq sup_n \|\bar{\lambda}^n P_+ bP_+ \lambda^n\| \leq \|P_+ bP_+\|$$

Commutators $[H, b]$

$[b, H]f = b \cdot Hf - H(bf)$ where H is the Hilbert transform and b is multiplication by the function b.

If we write
$H = P_+ - P_-$ and $I = P_+ + P_-$
then
$[b, H] = [(P_+ + P_-)b, (P_+ - P_-)] = 2P_- bP_+ - 2P_+ bP_-$,
two Hankel operators with orthogonal ranges.

Extensions of $[H, b]$

Riesz transform commutators and similar.

Coifman, Rochberg, Weiss, Uchiyama, Lacey ...

The passage to several parameters, initiation.

Cotlar, Ferguson, Sadosky ...

Thiele, Muscalu, Tao, Journe, Holmes, Lacey, Pipher, Strouse, Wick, P. ...

Commutators $[H_1 H_2, b]$ with $b(x_1, x_2)$

Tensor product one-parameter case.

Theorem (Ferguson, Sadosky)

$[H_1 H_2, b]$ *bounded in* L^2 *iff* $b \in bmo$ *'little BMO'*

$$||b||_{bmo} = \max\{\sup_{x_1} ||b(x_1, \cdot)||_{BMO}, \sup_{x_2} ||b(\cdot, x_2)||_{BMO}\}$$

i.e. b uniformly in BMO in each variable separately or of bounded mean oscillation on rectangles.

more Hilbert transforms $[H_1 H_2 H_3, b]$ etc implicit.

Commutators $[H_1,[H_2,b]]$ with $b(x_1,x_2)$

Theorem (Ferguson, Lacey)

$[H_1,[H_2,b]]$ bounded in L^2 iff $b \in BMO$ 'product BMO'

$$||b||^2_{BMO} = \sup_O \frac{1}{|O|} \sum_{R \subset O} |(b, h_R)|^2$$

more iterations $[H_1,[H_2,[H_3,b]]]$ 'not' implicit, but Terwilliger, Lacey.

Lower estimates for Hilbert commutators

Ferguson-Sadosky: elegant 'soft' argument, based on Toeplitz forms.

Ferguson-Lacey: extremely technical 'hard' real analysis argument based on Hankel forms. Using scale analysis, Schwarz tail estimates, geometric facts on distribution of rectangles in the plane...

Commutators $[H_2, [H_3 H_1, b]]$ with $b(x_1, x_2, x_3)$

Theorem (Ou, Strouse, P.)

$[H_2, [H_3 H_1, b]]$ *bounded in* L^2 *iff* $b \in BMO_{(13)2}$ *'little product BMO'*

$$||b||_{BMO_{(13)2}} = \max\{\sup_{x_1} ||b(x_1, \cdot, \cdot)||_{BMO}, \sup_{x_3} ||b(\cdot, \cdot, x_3)||_{BMO}\}$$

b uniformly in product BMO when fixing variables x_3 and x_1.
more Hilbert transforms and more iterations implicit.

Commutators $[H_2, [H_3 H_1, b]]$ with $b(x_1, x_2, x_3)$

Infact TFAE:

1. $b \in BMO_{(13)2}$
2. $[H_2, [H_1, b]]$ and $[H_2, [H_3, b]]$ bounded in $L^2(T^3)$
3. $[H_2, [H_3 H_1, b]]$ bounded in $L^2(T^3)$.

1 eq 2: Wiener's theorem and Ferguson, Lacey:

$$[H_2, [H_1, b]] f(x_1, x_2) g(x_3) = g(x_3) [H_2, [H_1, b]] f(x_1, x_2)$$

2 eq 3: Toeplitz argument: typical terms that arise:

$$P_1^+ P_2^+ b P_1^- P_2^- \text{ and } P_1^+ P_2^+ P_3^+ b P_1^- P_2^- P_3^+$$

Riesz

Riesz commutators

and the absence of Hankel and Toeplitz.

One parameter: $[R_i, b]$

It is a classical result by Coifman, Rochberg and Weiss, that the Riesz transform commutators classify BMO. For each symbol $b \in BMO$ we may choose the worst Riesz transform. In this sense

$$\|[b, R_i]\|_{2\to 2} \lesssim \|b\|_{BMO}$$

But

$$\|b\|_{BMO} \lesssim \sup_i \|[b, R_i]\|_{2\to 2}$$

Testing class for CZOs.

One parameter tensor product: $[R_{1,i_1}R_{2,i_2}, b]$

Through use of the little BMO norm, one sees that

$$\|[b, R_{1,i_1}R_{2,i_2}]\|_{2\to 2} \lesssim \|b\|_{bmo}$$

Through a direct calculation using the little BMO norm one also sees

$$\|b\|_{bmo} \lesssim \sup_{i_1,i_2} \|[b, R_{1,i_1}R_{2,i_2}]\|_{2\to 2}$$

multi-parameter: $[R_{2,j_2}, [R_{1,j_1}, b]]$

Theorem (Lacey, Pipher, Wick, P.)

$$\sup_{j_1, j_2} \|[R_{2,j_2}, [R_{1,j_1}, b]]\| \sim \|b\|_{BMO} \, .$$

By BMO, we mean Chang–Fefferman product BMO.

Implicit generalizations with similar proof.

Testing for CZOs. This means we test a symbol on Riesz transforms. The estimate then self improves to all operators of the same type as Riesz transforms.

multi-parameter tensor product: $[R_{2,j_2}, [R_{1,j_1} R_{3,j_3}, b]]$

Theorem (Ou, Strouse, P.)

$$\|b\|_{BMO_{(13)2}} \sim \|[R_{2,j_2}, [R_{1,j_1} R_{3,j_3}, b]]\|$$

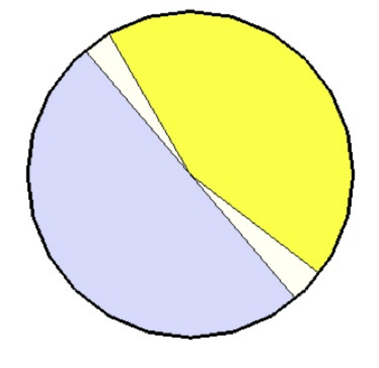
where we mean little product BMO.

Implicit generalizations but proof is substantially more difficult when dimensions are greater than 2 or when slots contain tensor products of more than 2.

Testing for (paraproduct-free) Journé operators - we will see later what these are.

Cones

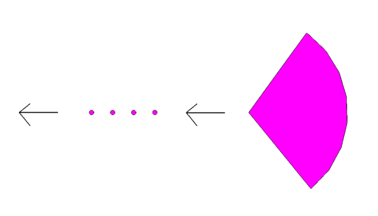
Riesz transforms do not have the same relation to projections as the Hilbert transform does.
Replace by well chosen, smooth half plane projections that are CZO.



Cones

Depending on the make of the symbol function, a multi parameter skeleton of cones of large aperture is chosen via a probabilistic procedure.

The others are filled in using tiny cones via a Toeplitz argument.



Polynomials

We have now showed that there is a lower estimate if one can choose from cone operators. How does this help for Riesz transforms?

Observe:

$$[T_1 T_2, b] = T_1[T_2, b] + [T_1, b]T_2$$

If the commutator with $T_1 T_2$ is large, then one of the commutators with T_i has to be large.

Riesz transforms have Fourier symbols ξ_i on $\mathbb{S}^n$ (monomials) well adapted for polynomial approximation.

Passage to tensor products of Riesz transforms

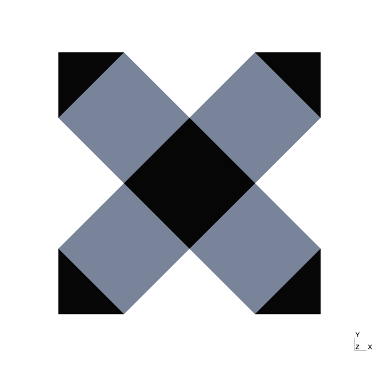
What goes wrong:

If $[R^2_{1,i_1} R_{2,i_2}, b]$ large, cannot say $[R_{1,i_1} R_{2,i_2}, b]$ remains large.

It is too wasteful to just have any lower estimates of tensor products of cone operators.

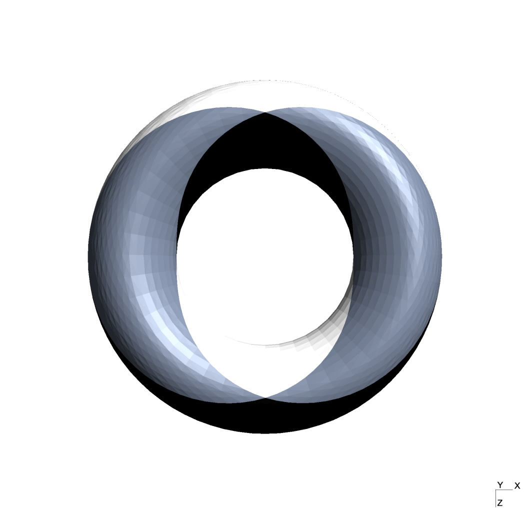
Passage to tensor products of Riesz transforms

These strip operators work well on products of $\mathbb{S}^1$ and a deep generalization using a probabilistic construction and zonal harmonics will work on higher dimensional spheres.



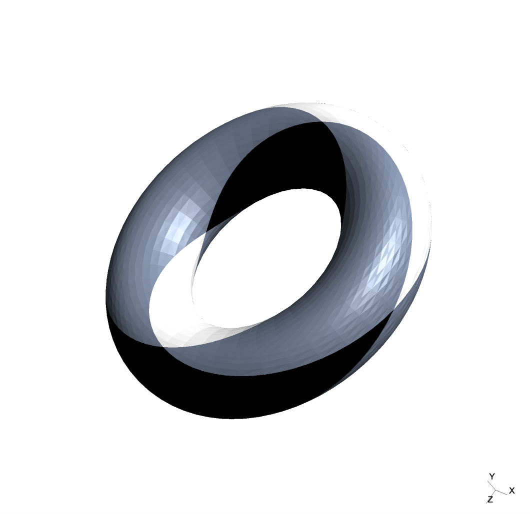
Passage to tensor products of Riesz transforms

Looking like this:



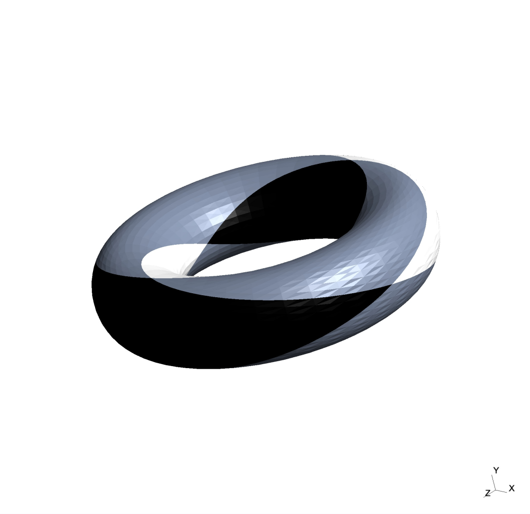
Passage to tensor products of Riesz transforms

and this:



Passage to tensor products of Riesz transforms

and this:



Upper estimates for Hilbert commutators

none of them are difficult (for the Hilbert transform), there are arguments using simple operator theory, but here is the proof that will extend.

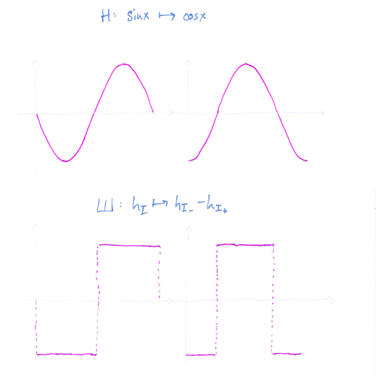
This simple operator is useful

$$S : h_I \mapsto h_{I_-} - h_{I_+}$$

The indexed intervals are dyadic intervals and h_I denotes the Haar basis.

The dyadic Hilbert transform

H gives access to harmonic conjugates, $z \mapsto z$ is analytic in $\mathbb{D}$ with boundary values $e^{it} = \cos(t) + i\sin(t)$.
So $'H(\sin) = \cos'$.



The dyadic Hilbert transform

The Hilbert transform can be written as expectation of dyadic Hilbert transforms. The proof is very elementary and explicit.
It turns out this is the right tool to capture cancellation in a commutator.

This tool was invented to address a question on commutators raised by Pisier. Its thrust lies in the immediate reduction of commutator bounds to so-called paraproducts. These are triangular sums in the naive multiplication

$$(\sum(f, h_I)h_I) \cdot (\sum(b, h_J)h_J)$$

Upper estimates

For the Riesz case need a stability estimate for Journe commutators to make the argument work:

$$||[J_1, ..., [J_t, b], ...,]|| \leq C||b||$$

with $C \to 0$ when defining constants of J do.

Use very general Haar shift operators for Journe operators that locally look like tensor products. (Hytonen, Martikainen)

Journé Operators

T defined on testing class $C_0^\infty \otimes C_0^\infty$ with two CZO kernels K_1, K_2 such that with $f_1(y_1), g_1(x_1), f_2(y_2)g_2(x_2)$

$$(T(f_1 \otimes f_2), g_1 \otimes g_2) = \int f_1(y_1)\langle K_1(x_1, y_1)f_2, g_2\rangle g_1(x_1)dx_1dy_1$$

if f_1, g_1 disjoint support

$$(T(f_1 \otimes f_2), g_1 \otimes g_2) = \int f_2(y_2)\langle K_2(x_2, y_2)f_1, g_1\rangle g_2(x_2)dx_2dy_2$$

if f_2, g_2 disjoint support weak boundedness property

$$|(T(1_I \otimes 1_J), 1_I \otimes 1_J) \lesssim |I||J|$$

Apply T to test function f_1 using first set of variables of the kernel. Get another CZO (depending on f_1) then applied to f_2.

Representation

Such operators have a representation using terms of the form

$$\sum \alpha_{I_1,J_1,K_1,I_2,J_2,K_2}(f, h_{I_1} \otimes h_{I_2}) h_{J_1} \otimes h_{J_2}$$

where the dyadic size difference of I_i and J_i to K_i is constant for each constellation. Then one sums in this difference and averages. There is a decay with this distance of the coefficients.

If there is a tensor product of CZO the coefficients split, but for true Journe operators, they are 'sticky'.

The matching mixed BMO condition is exactly correct to estimate commutators for these.