Bounded $H_\infty$-calculus for Closed Extensions of Cone Differential Operators

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Bounded $H_\infty$-calculus (sketch)

Throughout the talk, $\Lambda \subset \mathbb{C}$ is a closed sector with vertex in 0:

Let $A : \mathcal{D}(A) \subset X \to X$ be a sectorial operator, in particular,

$$\sup_{0 \neq \lambda \in \Lambda} \|\lambda (\lambda - A)^{-1}\|_{\mathcal{L}(X)} < +\infty.$$

**Definition:** $A$ admits a bounded $H_\infty$-calculus if

$$\|f(A)\|_{\mathcal{L}(X)} \leq c \|f\|_\infty$$

for all holomorphic and bounded $f : \mathbb{C} \setminus \Lambda \to \mathbb{C}$.
Bounded $H_\infty$-calculus (sketch)

**Remark:** For $f$ decaying to some $\varepsilon$-rate both at 0 and infinity,

$$f(A) = \frac{1}{2\pi i} \int_{\partial \Lambda} f(\lambda) (\lambda - A)^{-1} \, d\lambda \in \mathcal{L}(X).$$

**BUT:** For the estimate in terms of $\|f\|_\infty$, sectoriality is not enough. One needs “additional structure of the resolvent”.

**Approach:** If $A$ is a (pseudo-)differential operator, show that suitable **ellipticity assumptions** on $A$ imply that the resolvent has the structure of a **parameter-dependent pseudodifferential operator**.

**Note:** How the “ellipticity assumptions” and “pseudodifferential structure” look like depends heavily on what $A$ is.
Bounded $H_\infty$-calculus (advertisement)

- **Escher-S. (Trans. AMS 2005):**
  Pseudo’s of Hörmander-class $S_{1,\delta}^m$ with symbols of low regularity; in particular, the Dirichlet-Neumann-Operator for domains with $C^{1+\epsilon}$-boundary.

- **Denk-Saal-S. (Math. Nachr. 2009):**
  Douglis-Nirenberg systems (of low regularity).

- **Bilyj-Schrohe-S. (Proc. AMS 2010):**
  *Hypo*-elliptic pseudo’s from Weyl-Hörmander calculus.

  BIP and $H_\infty$-calculus for differential operators on manifolds with conic singularity.
Cone Differential Operators (for simplicity scalar)

Differential operators on the interior of a smooth compact manifold with boundary $\mathbb{B}$ with a specific "degenerate" structure near the boundary $X := \partial \mathbb{B}$:

In a collar-neighborhood $U \cong [0, \varepsilon) \times X$ of the boundary,

$$A = t^{-\mu} \sum_{j=0}^{\mu} a_j(t)(-t \partial_t)^j, \quad \mu = \text{ord } A,$$

with $a_j(t) \in \text{Diff}^{\mu-j}(X)$ depending smoothly on $t \in [0, \varepsilon)$.

Example (warped metric cone): The Laplacian with respect to a metric $g = dt^2 + t^2 g_X(t)$ is

$$\Delta = t^{-2} \left\{ (t \partial_t)^2 + (\dim X - 1 + a(t)) t \partial_t + \Delta_{X,t} \right\}$$

where $a(t) = t \partial_t (\log \det g_X(t))/2$. 


Weighted Sobolev spaces

$A$ acts in a scale of weighted Sobolev spaces

$$\mathcal{H}^{s, \gamma}_p(\mathbb{B}), \quad s, \gamma \in \mathbb{R}, \ 1 < p < +\infty$$

Definition ($s \in \mathbb{N}$): $u \in \mathcal{H}^{s, \gamma}_p(\mathbb{B})$ iff $u \in H^s_{p, loc}(\text{int } \mathbb{B})$ and

$$t^{\frac{n+1}{2} - \gamma} (t \partial_t)^j D^\alpha_x u(t, x) \in L^p\left(\mathbb{B}, \frac{dt}{t} dx\right), \quad j + |\alpha| \leq s.$$

Note: $s$ measures smoothness, $\gamma$ decay/growth-rate for $t \to 0$.

Note: $A$ of order $\mu$ induces continuous maps

$$A : \mathcal{H}^{s, \gamma}_p(\mathbb{B}) \rightarrow \mathcal{H}^{s-\mu, \gamma-\mu}_p(\mathbb{B})$$
Principal symbol(s) and conormal symbols

Principal symbol: $\sigma(A) \in C^\infty(T^* \text{int } B \setminus 0)$

Rescaled principal symbol: $\tilde{\sigma}(A) \in C^\infty((T^*X \times \mathbb{R}) \setminus 0)$ defined by

$$\tilde{\sigma}(A)(x, \xi; \tau) = \lim_{t \to 0} t^\mu \sigma(A)(t, x, \xi; t^{-1}\tau)$$

Conormal symbols: Operator-valued polynomials

$$h_k(z) = \frac{1}{k!} \sum_{j=0}^\mu \frac{d^k a_j}{dt^k}(0)z^j : \mathbb{C} \to \text{Diff}^\mu(X) \subset L^\mu_{\text{cl}}(X)$$

Ellipticity: $A$ elliptic with respect to $\gamma \in \mathbb{R}$ if

(a) (rescaled) principal symbol never vanishing,
(b) $h_0(z)$ invertible for every $z$ with $\Re z = \frac{n+1}{2} - \gamma$.

Note: (a) $\Rightarrow h_0(z)^{-1}$ meromorphic with values in $L^\mu_{\text{cl}}(X)$
Closed extensions of elliptic operators

Let $A$ be elliptic w.r.t. $\gamma + \mu$ and consider

$$A : \mathcal{C}_\text{comp}^\infty(\text{int } \mathbb{B}) \subset \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \longrightarrow \mathcal{H}_p^{s,\gamma}(\mathbb{B})$$

- **closure/minimal extension** given by
  $$\mathcal{D}_\text{min}(A) = \mathcal{H}_p^{s+\mu, \gamma+\mu}(\mathbb{B})$$

- **maximal extension** given by
  $$\mathcal{D}_\text{max}(A) = \mathcal{H}_p^{s+\mu, \gamma+\mu}(\mathbb{B}) \oplus \omega \mathcal{E}$$

where $\omega(t)$ is both $\equiv 1$ and supported near the boundary, and $\mathcal{E}$ is a finite-dimensional space of smooth functions not depending on $s$ and $p$.

- **An arbitrary closed extension** $\mathcal{A}$ of $A$ is given by a domain
  $$\mathcal{D}(\mathcal{A}) = \mathcal{H}_p^{s+\mu, \gamma+\mu}(\mathbb{B}) \oplus \mathcal{E}, \quad \mathcal{E} \subset \mathcal{E}$$
The space $\mathcal{E} \cong \mathcal{D}_{\text{max}}(A)/\mathcal{D}_{\text{min}}(A)$

**Definition:** The model-cone operator associated with $A$ is

$$\hat{A} = t^{-\mu} \sum_{j=0}^{\mu} a_j(0)(-t\partial_t)^j$$

It is a differential operator on $X^\wedge := (0, +\infty) \times X$. Let

$$\hat{\mathcal{E}} = \text{span}\left\{ u = t^{-p} \sum_k c_k(x) \ln^k t \mid \hat{A}u = 0, \frac{n+1}{2} - \text{Re} p \in (\gamma, \gamma + \mu) \right\}$$

(determined by the poles of $h_0(z)^{-1}$ with $\frac{n+1}{2} - \text{Re} z \in (\gamma, \gamma + \mu)$).

**Proposition (Gil-Krainer-Mendoza 2006, S. 2010):** $\mathcal{E}$ is determined by $h_0(z)^{-1}$ and $h_1(z), \ldots, h_\mu(z)$. It has the same dimension as $\hat{\mathcal{E}}$ and there is a canonical isomorphism

$$\Theta : \text{Gr}(\hat{\mathcal{E}}) \longrightarrow \text{Gr}(\mathcal{E})$$

(Grassmannians)
Example: Laplacian in dimension 2

\[ \Delta = t^{-2} \left( (t \partial_t)^2 + a(t)(t \partial_t) + \Delta_{t,x} \right) \quad \text{in} \]

Conormal symbols: \( h_0(z) = z^2 + \Delta_0 \), \( h_1(z) = -\dot{a}(0)z + \dot{\Delta}_0 \)

Poles of \( h_0(z)^{-1} \): 0 double pole, \( \pm \sqrt{-\lambda_j} \) simple poles

Passage from \( \hat{\mathcal{E}} \) to \( \mathcal{E} \): Assume \( -\lambda_j > 1 \). The function

\[ c_0 + c_1 \ln t \in \hat{\mathcal{E}}, \quad c_0, c_1 \in \mathbb{C}, \]

generates

\[ c_0 + c_1(\ln t + t \ c(x)) \in \mathcal{E}, \quad c(\cdot) = h_0(-1)^{-1} \dot{a}(0). \]
Parameter-ellipticity: The minimal extension

The minimal extension falls into Schulze’s calculus of parameter-dependent cone pseudodifferential operators (“cone algebra”):

1. Both $\sigma(A)$ and $\tilde{\sigma}(A)$ do not take values in $\Lambda$,
2. $A$ is elliptic w.r.t. $\gamma + \mu$,
3. $\hat{A} : \mathcal{K}_2^{\mu,\gamma+\mu}(X^\wedge) \subset \mathcal{K}_2^{0,\gamma}(X^\wedge) \rightarrow \mathcal{K}_2^{0,\gamma}(X^\wedge)$ has no spectrum in $\Lambda \setminus 0$.

Note: (3) is a kind of “Shapiro-Lopatinskij condition”

**Theorem (Schulze):** In this case, there exists a $c \geq 0$ such that $A_{\text{min}} + c$ in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ is sectorial and its resolvent has a certain pseudodifferential structure.

**Theorem (Coriasco-Schrohe-S. 03):** In this case, $A_{\text{min}} + c$ has BIP in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$. 
Parameter-ellipticity: Scaling invariant extensions

Let $A$ have $t$-independent coefficients. Let $\overline{A}$ have a domain

$$\mathcal{D}(\overline{A}) = \mathcal{H}_{p}^{\mu,\gamma+\mu}(\mathbb{B}) + \omega \mathcal{E}$$

Assumptions:

- $\mathcal{E}$ is invariant under dilations:
  $$u(t, x) \in \mathcal{E} \Rightarrow u(st, x) \in \mathcal{E} \quad \forall \ s > 0.$$

- $A$ satisfies (1), (2) from above and

  (3) $\hat{A}: \mathcal{K}_{2}^{\mu,\gamma+\mu}(X^{\wedge}) \oplus \omega \mathcal{E} \subset \mathcal{K}_{2}^{0,\gamma}(X^{\wedge}) \rightarrow \mathcal{K}_{2}^{0,\gamma}(X^{\wedge})$

  has no spectrum in $\Lambda \setminus \{0\}$.

Theorem (Schrohe-S. 05): In this case, there exists a $c \geq 0$ such that $\overline{A} + c$ in $\mathcal{H}_{p}^{0,\gamma}(\mathbb{B})$ is sectorial and its resolvent has a certain pseudodifferential structure. Moreover, $\overline{A} + c$ has BIP.
Parameter-ellipticity: General extensions

Theorem (Schrohe-S. 2005/07): Let $\underline{A}$ be a closed extension of $A$ in $\mathcal{H}^{0,\gamma}_{p}(\mathbb{B})$. Assume that the resolvent exists and has a certain pseudodifferential structure. Then $\underline{A}$ has a bounded $H_{\infty}$-calculus.

Theorem (Schrohe-Roidos 2014): The previous theorem remains true for extensions $\underline{A}$ of $A$ in $\mathcal{H}^{s,\gamma}_{p}(\mathbb{B})$, $s \geq 0$.

Theorem (Gil-Krainer-Mendoza 2006): Let $A$ satisfy (1), (2) and let $\underline{A}$ be an extension in $\mathcal{H}^{0,\gamma}_{2}(\mathbb{B})$ such that

\begin{equation}
\hat{A} : \mathcal{K}^{\mu,\gamma+\mu}_{2}(X^\wedge) \oplus \omega(\Theta^{-1} \Theta) \subset \mathcal{K}^{0,\gamma}_{2}(X^\wedge) \longrightarrow \mathcal{K}^{0,\gamma}_{2}(X^\wedge)
\end{equation}

is invertible for large $\lambda \in \Lambda$ with $\|\lambda(\lambda - \hat{A})^{-1}\|$ uniformly bounded.

Then there exists a $c \geq 0$ such that $\underline{A} + c$ is sectorial in $\mathcal{H}^{0,\gamma}_{2}(\mathbb{B})$.

Note: Resolvent has a slightly different pseudodifferential structure.
Theorem (Schrohe-S. 2017): Let \( A \) be an extension of \( A \) in \( \mathcal{H}_{p}^{s, \gamma}(\mathbb{B}) \), \( s \geq 0 \), such that

1. Both \( \sigma(A) \) and \( \tilde{\sigma}(A) \) do not take values in \( \Lambda \),
2. \( A \) is elliptic with respect to \( \gamma + \mu \) and \( \gamma \),
3. Gil-Krainer-Mendoza’s condition on the model-cone operator \( \hat{A} \) holds true.

Then there exists a \( c \geq 0 \) such that \( \underline{A} + c \) is sectorial, the resolvent has a certain pseudodifferential structure, and \( \underline{A} + c \) has a bounded \( H_{\infty} \)-calculus.

Note: Condition (2) means that \( A \) and \( A^* \) are elliptic w.r.t. \( \gamma + \mu \).
The pseudodifferential structure

Fourier transform: \( D_\alpha x \longrightarrow \xi^\alpha \)
Mellin transform: \((-t\partial_t)^k \longrightarrow z^k\)

Rough idea: The resolvent is of the form

\[
(\lambda - A)^{-1} = t^\mu H(\lambda) + P(\lambda) + G(\lambda)
\]

- \(H(\lambda)\) parameter-dependent Mellin pseudo of order \(-\mu\) with holomorphic symbol, supported near the boundary;
- \(P(\lambda)\) parameter-dependent Fourier pseudo of order \(-\mu\), supported away from the boundary;
- \(G(\lambda)\) parameter-dependent Green operator (smoothing).

Note: • \(t^\mu H(\lambda) + P(\lambda)\) maps into \(\mathcal{H}_{p}^{s+\mu,\gamma+\mu}(B)\)
  • \(G(\lambda)\) “generates” \(\mathcal{C}\).
Parameter-dependent Green operators

Fact: $\mathcal{D}_{\text{max}}(A) \subset \mathcal{H}_p^{s+\mu, \gamma+\varepsilon}(\mathbb{B})$ for some $\varepsilon > 0$.

Green operators: Are of the form

$$G(\lambda) = \omega K(\lambda) \omega + R(\lambda)$$

- $R(\lambda)$ is an integral operator with smooth kernel $r(y, y'; \lambda)$ vanishing at $\partial\mathbb{B}$ to order $\gamma + \varepsilon$ in $y$, to order $-\gamma + \varepsilon$ in $y'$, and vanishes of infinite order for $|\lambda| \rightarrow +\infty$;

- $K(\lambda)$ is an integral operator on $X^\wedge$ with smooth kernel

$$k(t, x, t', x'; \lambda) = \tilde{k}(t[\lambda]^{1/\mu}, x, t'[\lambda]^{1/\mu}, x'; \lambda)$$

where $\tilde{k}(s, x, s', x'; \lambda)$ vanishes at $s = 0/s' = 0$ of rate $\gamma + \varepsilon/ -\gamma + \varepsilon$ and at $s = +\infty/s' = +\infty$ of infinite order, and behaves in $\lambda$ as a pseudodifferential symbol of order $-1$. 

Bounded $H_\infty$-calculus

Estimate the Dunford-integral for $f(A)$ using the above structure of the resolvent:

- $P(\lambda)$ and $t^{\mu}H(\lambda)$ produce Fourier/Mellin pseudo’s of order 0 with symbol estimates involving only $\|f\|_\infty$.
- $G(\lambda)$ is treated using a certain Hardy integral inequality.
Thank you for your attention!
Vielen Dank für Ihre Aufmerksamkeit!
Grazie per la vostra attenzione!