Bounded H_∞-calculus for Closed Extensions of Cone Differential Operators

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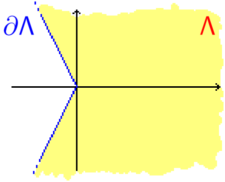
Bounded H_∞-calculus (sketch)

Throughout the talk, $\Lambda \subset \mathbb{C}$ is a closed sector with vertex in 0 :



Let $A : \mathscr{D}(A) \subset X \to X$ be a sectorial operator, in particular,

$$\sup_{0 \neq \lambda \in \Lambda} \|\lambda(\lambda - A)^{-1}\|_{\mathscr{L}(X)} < +\infty.$$

Definition: A admits a bounded H_∞-calculus if

$$\|f(A)\|_{\mathscr{L}(X)} \leq c\,\|f\|_\infty$$

for all holomorphic and bounded $f : \mathbb{C} \setminus \Lambda \to \mathbb{C}$.

Bounded H_∞-calculus (sketch)

Remark: For f decaying to some ε-rate both at 0 and infinity,

$$\boxed{f(A) = \frac{1}{2\pi i}\int_{\partial\Lambda} f(\lambda)\,(\lambda - A)^{-1}\,d\lambda} \in \mathscr{L}(X).$$

BUT: For the estimate in terms of $\|f\|_\infty$, sectoriality is not enough. One needs "additional structure of the resolvent".

Approach: If A is a (pseudo-)differential operator, show that suitable ellipticity asumptions on A imply that the resolvent has the structure of a parameter-dependent pseudodifferential operator.

Note: How the "ellipticity asumptions" and "pseudodifferential structure" look like depends heavily on what A is.

Bounded H_∞-calculus (advertisement)

- Escher-S. (Trans. AMS 2005):
  Pseudo's of Hörmander-class $S^m_{1,\delta}$ with symbols of low regularity; in particular, the Dirichlet-Neumann-Operator for domains with $\mathscr{C}^{1+\varepsilon}$-boundary.
- Denk-Saal-S. (Math. Nachr. 2009):
  Douglis-Nirenberg systems (of low regularity).
- Bilyj-Schrohe-S. (Proc. AMS 2010):
  Hypo-elliptic pseudo's from Weyl-Hörmander calculus.
- Coriasco/Schrohe-S. (Math. Z. 2003, Canad. J. Math. 2005, Comm. PDE 2007, Preprint 2017):
  BIP and H_∞-calculus for differential operators on manifolds with conic singularity.

Cone Differential Operators (for simplicity scalar)

Differential operators on the interior of a smooth compact manifold with boundary $\mathbb{B}$ with a specific "degenerate" structure near the boundary $X := \partial\mathbb{B}$:

In a collar-neighborhood $U \cong [0,\varepsilon) \times X$ of the boundary,

$$A = t^{-\mu} \sum_{j=0}^{\mu} a_j(t)(-t\partial_t)^j, \qquad \mu = \operatorname{ord} A,$$

with $a_j(t) \in \mathrm{Diff}^{\mu-j}(X)$ depending smoothly on $t \in [0,\varepsilon)$.

Example (warped metric cone): The Laplacian with respect to a metric $g = dt^2 + t^2 g_X(t)$ is

$$\Delta = t^{-2}\Big\{(t\partial_t)^2 + (\dim X - 1 + a(t))\, t\partial_t + \Delta_{X,t}\Big\}$$

where $a(t) = t\partial_t(\log \det g_X(t))/2$.

Weighted Sobolev spaces

A acts in a scale of weighted Sobolev spaces

$$\mathcal{H}_p^{s,\gamma}(\mathbb{B}), \qquad s,\gamma \in \mathbb{R},\ 1 < p < +\infty$$

Definition ($s \in \mathbb{N}$): $u \in \mathcal{H}_p^{s,\gamma}(\mathbb{B})$ iff $u \in H^s_{p,loc}(\operatorname{int}\mathbb{B})$ and

$$t^{\frac{n+1}{2}-\gamma}(t\partial_t)^j D_x^\alpha u(t,x) \in L^p\Big(\mathbb{B}, \frac{dt}{t}dx\Big), \qquad j + |\alpha| \le s.$$

Note: s measures smoothness, γ decay/growth-rate for $t \to 0$.

Note: A of order μ induces continuous maps

$$A : \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \longrightarrow \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B})$$

Principal symbol(s) and conormal symbols

Principal symbol: $\sigma(A) \in \mathscr{C}^\infty(T^*\mathrm{int}\,\mathbb{B} \setminus 0)$

Rescaled principal symbol: $\widetilde{\sigma}(A) \in \mathscr{C}^\infty((T^*X \times \mathbb{R}) \setminus 0)$ defined by

$$\widetilde{\sigma}(A)(x,\xi;\tau) = \lim_{t \to 0} t^\mu \, \sigma(A)(t,x,\xi;t^{-1}\tau)$$

Conormal symbols: Operator-valued polynomials

$$h_k(z) = \frac{1}{k!} \sum_{j=0}^{\mu} \frac{d^k a_j}{dt^k}(0) z^j : \mathbb{C} \longrightarrow \mathrm{Diff}^\mu(X) \subset L^\mu_{\mathrm{cl}}(X)$$

Ellipticity: A elliptic with respect to $\gamma \in \mathbb{R}$ if

(a) (rescaled) principal symbol never vanishing,

(b) $h_0(z)$ invertible for every z with $\mathrm{Re}\, z = \frac{n+1}{2} - \gamma$.

Note: (a) $\Rightarrow$ $h_0(z)^{-1}$ meromorphic with values in $L^{-\mu}_{\mathrm{cl}}(X)$

Closed extensions of elliptic operators

Let A be elliptic w.r.t. $\gamma + \mu$ and consider

$$A : \mathscr{C}^{\infty}_{\mathrm{comp}}(\mathrm{int}\,\mathbb{B}) \subset \mathcal{H}^{s,\gamma}_p(\mathbb{B}) \longrightarrow \mathcal{H}^{s,\gamma}_p(\mathbb{B})$$

- closure/minimal extension given by $\mathscr{D}_{\min}(A) = \mathcal{H}^{s+\mu,\gamma+\mu}_p(\mathbb{B})$
- maximal extension given by

  $$\mathscr{D}_{\max}(A) = \mathcal{H}^{s+\mu,\gamma+\mu}_p(\mathbb{B}) \oplus \omega\mathscr{E}$$

  where $\omega(t)$ is both $\equiv 1$ and supported near the boundary, and $\mathscr{E}$ is a finite-dimensional space of smooth functions not depending on s and p.
- An arbitrary closed extension $\underline{A}$ of A is given by a domain

  $$\mathscr{D}(\underline{A}) = \mathcal{H}^{s+\mu,\gamma+\mu}_p(\mathbb{B}) \oplus \underline{\mathscr{E}}, \qquad \underline{\mathscr{E}} \subset \mathscr{E}$$

The space $\mathscr{E} \cong \mathscr{D}_{\max}(A)/\mathscr{D}_{\min}(A)$

Definition: The model-cone operator associated with A is

$$\widehat{A} = t^{-\mu} \sum_{j=0}^{\mu} a_j(0)(-t\partial_t)^j$$

It is a differential operator on $X^\wedge := (0, +\infty) \times X$. Let

$$\widehat{\mathscr{E}} = \operatorname{span}\Big\{ u = t^{-p} \sum_k c_k(x) \ln^k t \mid \widehat{A}u = 0,\ \tfrac{n+1}{2} - \operatorname{Re} p \in (\gamma, \gamma+\mu) \Big\}$$

(determined by the poles of $h_0(z)^{-1}$ with $\frac{n+1}{2} - \operatorname{Re} z \in (\gamma, \gamma+\mu)$).

Proposition (Gil-Krainer-Mendoza 2006, S. 2010): $\mathscr{E}$ is determined by $h_0(z)^{-1}$ and $h_1(z), \ldots, h_\mu(z)$. It has the same dimension as $\widehat{\mathscr{E}}$ and there is a canonical isomorphism

$$\Theta : \operatorname{Gr}(\widehat{\mathscr{E}}) \longrightarrow \operatorname{Gr}(\mathscr{E}) \qquad \text{(Grassmannians)}$$

Example: Laplacian in dimension 2

$$\Delta = t^{-2}\Big((t\partial_t)^2 + a(t)(t\partial_t) + \Delta_{t,X}\Big) \quad \text{in}$$

Conormal symbols: $h_0(z) = z^2 + \Delta_0, \quad h_1(z) = -\dot{a}(0)z + \dot{\Delta}_0$

Poles of $h_0(z)^{-1}$: 0 double pole, $\pm\sqrt{-\lambda_j}$ simple poles

Passage from $\widehat{\mathscr{E}}$ to $\mathscr{E}$: Assume $-\lambda_j > 1$. The function

$$c_0 + c_1 \ln t \in \widehat{\mathscr{E}}, \qquad c_0, c_1 \in \mathbb{C},$$

generates

$$c_0 + c_1(\ln t + t\, c(x)) \in \mathscr{E}, \qquad c(\cdot) = h_0(-1)^{-1}\dot{a}(0).$$

Parameter-ellipticity: The minimal extension

The minimal extension falls into **Schulze's calculus** of parameter-dependent cone pseudodifferential operators ("cone algebra"):

(1) Both $\sigma(A)$ and $\widetilde{\sigma}(A)$ do not take values in Λ,

(2) A is elliptic w.r.t. $\gamma + \mu$,

(3) $\widehat{A} : \mathcal{K}_2^{\mu,\gamma+\mu}(X^\wedge) \subset \mathcal{K}_2^{0,\gamma}(X^\wedge) \longrightarrow \mathcal{K}_2^{0,\gamma}(X^\wedge)$ has no spectrum in $\Lambda \setminus 0$.

Note: (3) is a kind of "Shapiro-Lopatinskij condition"

Theorem (Schulze): In this case, there exists a $c \geq 0$ such that $A_{\min} + c$ in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ is sectorial and its resolvent has a certain pseudodifferential structure.

Theorem (Coriasco-Schrohe-S. 03): In this case, $A_{\min} + c$ has BIP in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$.

Parameter-ellipticity: Scaling invariant extensions

Let A have t-independent coefficients. Let $\underline{A}$ have a domain

$$\mathscr{D}(\underline{A}) = \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{B}) + \omega\underline{\mathscr{E}}$$

Assumptions:

- $\underline{\mathscr{E}}$ is invariant under dilations:

$$u(t,x) \in \underline{\mathscr{E}} \Rightarrow u(st,x) \in \underline{\mathscr{E}} \quad \forall\, s > 0.$$

- A satisfies (1), (2) from above and

  (3) $\widehat{A} : \mathcal{K}_2^{\mu,\gamma+\mu}(X^\wedge) \oplus \omega\underline{\mathscr{E}} \subset \mathcal{K}_2^{0,\gamma}(X^\wedge) \longrightarrow \mathcal{K}_2^{0,\gamma}(X^\wedge)$
  has no spectrum in $\Lambda \setminus 0$.

Theorem (Schrohe-S. 05): In this case, there exists a $c \geq 0$ such that $\underline{A} + c$ in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ is sectorial and its resolvent has a certain pseudodifferential structure. Moreover, $\underline{A} + c$ has BIP.

Parameter-ellipticity: General extensions

Theorem (Schrohe-S. 2005/07): Let $\underline{A}$ be a closed extension of A in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$. **Assume** that the resolvent exists and has a certain pseudodifferential structure. Then $\underline{A}$ has a bounded H_∞-calculus.

Theorem (Schrohe-Roidos 2014): The previous theorem remains true for extensions $\underline{A}$ of A in $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$, $s \geq 0$.

Theorem (Gil-Krainer-Mendoza 2006): Let A satisfy (1), (2) and let $\underline{A}$ be an extension in $\mathcal{H}_2^{0,\gamma}(\mathbb{B})$ such that

(3) $\widehat{A} : \mathcal{K}_2^{\mu,\gamma+\mu}(X^\wedge) \oplus \omega(\Theta^{-1}\underline{\mathscr{E}}) \subset \mathcal{K}_2^{0,\gamma}(X^\wedge) \longrightarrow \mathcal{K}_2^{0,\gamma}(X^\wedge)$

is invertible for large $\lambda \in \Lambda$ with $\|\lambda(\lambda - \widehat{A})^{-1}\|$ uniformly bounded.

Then there exists a $c \geq 0$ such that $\underline{A} + c$ is sectorial in $\mathcal{H}_2^{0,\gamma}(\mathbb{B})$.

Note: Resolvent has a slightly different pseudodifferential structure.

Parameter-ellipticity: General extensions

Theorem (Schrohe-S. 2017): Let $\underline{A}$ be an extension of A in $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$, $s \geq 0$, such that

(1) Both $\sigma(A)$ and $\widetilde{\sigma}(A)$ do not take values in Λ,

(2) A is elliptic with respect to $\gamma + \mu$ and γ,

(3) Gil-Krainer-Mendoza's condition on the model-cone operator $\widehat{A}$ holds true.

Then there exists a $c \geq 0$ such that $\underline{A} + c$ is sectorial, the resolvent has a certain pseudodifferential structure, and $\underline{A} + c$ has a bounded H_∞-calculus.

Note: Condition (2) means that A and A^* are elliptic w.r.t. $\gamma + \mu$

The pseudodifferential structure

Fourier transform: $D_x^\alpha \longrightarrow \xi^\alpha$

Mellin transform: $(-t\partial_t)^k \longrightarrow z^k$

Rough idea: The resolvent is of the form

$$(\lambda - \underline{A})^{-1} = t^\mu H(\lambda) + P(\lambda) + G(\lambda)$$

- $H(\lambda)$ parameter-dependent Mellin pseudo of order $-\mu$ with holomorphic symbol, supported near the boundary;
- $P(\lambda)$ parameter-dependent Fourier pseudo of order $-\mu$, supported away from the boundary;
- $G(\lambda)$ parameter-dependent Green operator (smoothing).

Note: • $t^\mu H(\lambda) + P(\lambda)$ maps into $\mathcal{H}_p^{s+\mu,\gamma+\mu}(\mathbb{B})$

• $G(\lambda)$ "generates" $\underline{\mathscr{E}}$.

Parameter-dependent Green operators

Fact: $\mathscr{D}_{\max}(A) \subset \mathcal{H}_p^{s+\mu,\gamma+\varepsilon}(\mathbb{B})$ for some $\varepsilon > 0$.

Green operators: Are of the form

$$G(\lambda) = \omega K(\lambda)\,\omega + R(\lambda)$$

- $R(\lambda)$ is an integral operator with smooth kernel $r(y, y'; \lambda)$ vanishing at $\partial\mathbb{B}$ to order $\gamma + \varepsilon$ in y, to order $-\gamma + \varepsilon$ in y', and vanishes of infinite order for $|\lambda| \to +\infty$;
- $K(\lambda)$ is an integral operator on $X^\wedge$ with smooth kernel

$$k(t, x, t', x'; \lambda) = \widetilde{k}(t[\lambda]^{1/\mu}, x, t'[\lambda]^{1/\mu}, x'; \lambda)$$

  where $\widetilde{k}(s, x, s', x'; \lambda)$ vanishes at $s = 0/s' = 0$ of rate $\gamma + \varepsilon / -\gamma + \varepsilon$ and at $s = +\infty/s' = +\infty$ of infinite order, and behaves in λ as a pseudodifferential symbol of order -1.

Bounded H_∞-calculus

Estimate the Dunford-integral for $f(\underline{A})$ using the above structure of the resolvent:

- $P(\lambda)$ and $t^\mu H(\lambda)$ produce Fourier/Mellin pseudo's of order 0 with symbol estimates involving only $\|f\|_\infty$,
- $G(\lambda)$ is treated using a certain Hardy integral inequality.

Thank you for your attention !

Vielen Dank für Ihre Aufmerksamkeit !

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