Generalized Solutions of Riccati equations and inequalities

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Time-invariant system

Time-invariant system with discrete time n

$$\Sigma \begin{cases} x_{n+1} &= Ax_n + Bu_n \\ y_n &= Cx_n + Du_n \end{cases}$$

$$A : \mathcal{X} \to \mathcal{X}, \quad B : \mathcal{U} \to \mathcal{X},$$
$$C : \mathcal{X} \to \mathcal{Y}, \quad D : \mathcal{U} \to \mathcal{Y}$$

A, B, C, D are bounded linear operators between Hilbert spaces.

Starting at time *0* with initial state x_0 and input $u_0, u_1, u_2, \ldots$
we compute the output $y_0, y_1, y_2, \ldots$

$$y_k = CA^k x_0 + \sum_{j=0}^{k-1} CA^{k-1-j} B u_j + D u_k.$$

$$\Sigma \begin{cases} x_{n+1} & = & Ax_n + Bu_n \\ y_n & = & Cx_n + Du_n \end{cases}$$

$$y_k = CA^k x_0 + \sum_{j=0}^{k-1} CA^{k-1-j} B u_j + D u_k.$$

Transfer function

$$\theta_\Sigma(\lambda) = D + \sum_{j\geq 1} CA^{j-1} B \lambda^j = D + \lambda C \left(I - \lambda A\right)^{-1} B$$

Starting at time *0* with initial state ***$x_0 = 0$*** and input ***$u_0, u_1, u_2, \ldots$***
we compute the output ***$y_0, y_1, y_2, \ldots$*** by multiplication

$$u(\lambda) = \sum_{j\geq 1} u_j \lambda^j$$

$$y(\lambda) = \sum_{j\geq 1} y_j \lambda^j$$

$$\theta(\lambda) u(\lambda) = y(\lambda)$$

A system $\Sigma \left\{ \begin{array}{rcl} x_{n+1} & = & Ax_n + Bu_n \\ y_n & = & Cx_n + Du_n \end{array} \right.$ is called a ***realization*** of θ if

$\theta_\Sigma(\lambda) = \theta(\lambda)$ in a neighborhood of *0*.

Two fundamental subspaces of the state space

$$\text{Im } (A|B) = \text{span }_{n\geq 0}\text{Im } A^n B$$

$$\text{Ker } (C|A) = \bigcap_{n\geq 0} \text{Ker } CA^n$$

The system Σ is ***controllable*** if $\overline{\text{Im } (A|B)} = \mathcal{X}$

The system Σ is ***observable*** if $\text{Ker } (C|A) = \{0\}$

The system $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}, D; \tilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y})$ is a ***dilation*** of the system

$\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$

if $\tilde{\mathcal{X}} = E \oplus \mathcal{X} \oplus E_*$ such that

$$\tilde{\Sigma} = \left(\begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & \boxed{A} & A_5 \\ 0 & 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ \boxed{B} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & \boxed{C} & C_1 \end{bmatrix}, D; E \oplus \mathcal{X} \oplus E_*, \mathcal{U}, \mathcal{Y} \right)$$

The system Σ is a ***restriction*** of $\tilde{\Sigma}$.

A system is ***minimal*** if it is not a dilation of any other (different) system.

Prop. A system is minimal iff it is controllable and observable.

The system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is called ***passive***

if for each initial condition x_0 and input sequence $u_0, u_1, u_2, \ldots$

$$\|x_{n+1}\|^2 - \|x_n\|^2 \leq \|u_n\|^2 - \|y_n\|^2$$

$\Leftrightarrow$ The system matrix $M_\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{X} \oplus \mathcal{U} \to \mathcal{X} \oplus \mathcal{Y}$ is a contraction.

Two theorems

$\|M_\Sigma\| \leq 1 \quad \Rightarrow \quad \theta(\cdot)$ is a Schur class function

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$\theta(\cdot)$ is a Schur class function $\quad \Rightarrow \quad \theta(\cdot)$

- appears as the transfer function of a unitary system [Br, NF]
- appears as the transfer function of a minimal and passive system.

Finite dimensions

Consider a rational $\mathbb{C}^{q\times p}$ -valued function θ , analytic in a neighborhood of 0,

and let $\Sigma = (A, B, C, D; \mathbb{C}^n, \mathbb{C}^p, \mathbb{C}^q)$ be a minimal realization of θ .

State space similarity theorem: all minimal realizations of θ are given by

$$\Sigma(S) = (SAS^{-1}, SB, CS^{-1}, D; \mathbb{C}^n, \mathbb{C}^p, \mathbb{C}^q)$$

where $S \in \mathbb{C}^{n\times n}$ is an invertible matrix.

Kalman-Yakubovich-Popov Lemma

Given a rational Schur class function with minimal realization

$$\theta_\Sigma(\lambda) = D + \lambda C\,(I - \lambda A)^{-1}\,B$$

Then there exists an invertible $S \in \mathbb{C}^{n\times n}$ such that

$\Sigma(S) = (SAS^{-1}, SB, CS^{-1}, D; \mathbb{C}^n, \mathbb{C}^p, \mathbb{C}^q)$ is passive.

This implies that for $H = S^*S$:

$$\begin{bmatrix} H - A^*HA - C^*C & -C^*D - A^*HB \\ -D^*C - B^*HA & I - D^*D - B^*HB \end{bmatrix} \geq 0$$

In this case: A is stable.

Conversely: if A is stable and $H > 0$ satisfies the above inequality, then $\Sigma(H^{1/2})$ is a passive system and θ is in the Schur class.

Schur complement

Let Σ be a minimal system and θ a rational Schur class function.

We want to find positive and invertible *H* such that

$$\begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix} = \begin{bmatrix} H - A^*HA - C^*C & -C^*D - A^*HB \\ -D^*C - B^*HA & I - D^*D - B^*HB \end{bmatrix} \geq 0$$

Schur complement

$$\begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix} \overset{?}{=} \begin{bmatrix} I & \beta\delta^{[-1]} \\ 0 & I \end{bmatrix} \begin{bmatrix} \alpha - \beta\delta^{[-1]}\beta^* & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} I & 0 \\ \delta^{[-1]}\beta^* & I \end{bmatrix}$$

Moore-Penrose inverse: $\delta^{[-1]}$

Moore Penrose Inverse

Self-adjoint matrix $\delta \in \mathbb{C}^{p\times p}$

Put $\mathcal{X}_1 = \operatorname{Im} \delta$ $\mathcal{X}_2 = \operatorname{Ker} \delta$ and $\delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{X}_1 \oplus \mathcal{X}_2 \to \mathcal{X}_1 \oplus \mathcal{X}_2$

Then the Moore Penrose Inverse is defined by

$$\delta^{[-1]} = \begin{bmatrix} \delta_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{X}_1 \oplus \mathcal{X}_2 \to \mathcal{X}_1 \oplus \mathcal{X}_2$$

$$\delta \cdot \delta^{[-1]} = P_{\operatorname{Im} \delta} \qquad \delta^{[-1]} \cdot \delta = P_{\operatorname{Im} \delta}$$

Schur complement

Let Σ be a minimal system and θ a rational Schur class function.

We want to find positive and invertible H such that

$$\begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix} = \begin{bmatrix} H - A^*HA - C^*C & -C^*D - A^*HB \\ -D^*C - B^*HA & I - D^*D - B^*HB \end{bmatrix} \geq 0$$

Schur complement

$$\begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix} = \begin{bmatrix} I & \beta\delta^{[-1]} \\ 0 & I \end{bmatrix} \begin{bmatrix} \alpha - \beta\delta^{[-1]}\beta^* & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} I & 0 \\ \delta^{[-1]}\beta^* & I \end{bmatrix}$$

Moore-Penrose inverse: $\delta^{[-1]}$ $\quad \delta \cdot \delta^{[-1]} = P_{\mathrm{Im}\,\delta}$

Condition: $\mathrm{Im}\,\beta^* \subset \mathrm{Im}\,\delta$

We want to find positive and invertible H such that

$$\begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix} = \begin{bmatrix} H - A^*HA - C^*C & -C^*D - A^*HB \\ -D^*C - B^*HA & I - D^*D - B^*HB \end{bmatrix} \geq 0$$

Schur complement
$$\begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix} = \begin{bmatrix} I & \beta\delta^{[-1]} \\ 0 & I \end{bmatrix} \begin{bmatrix} \alpha - \beta\delta^{[-1]}\beta^* & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} I & 0 \\ \delta^{[-1]}\beta^* & I \end{bmatrix}$$

Condition: $\operatorname{Im} \beta^* \subset \operatorname{Im} \delta$

$$\begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix} \geq 0 \quad \Leftrightarrow \quad \begin{cases} \operatorname{Im} \beta^* \subset \operatorname{Im} \delta \\ \alpha - \beta\delta^{[-1]}\beta^* \geq 0 \\ \delta \geq 0 \end{cases}$$

$$\begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix} = \begin{bmatrix} H - A^*HA - C^*C & -C^*D - A^*HB \\ -D^*C - B^*HA & I - D^*D - B^*HB \end{bmatrix} \geq 0$$

$$\begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix} \geq 0 \quad \Leftrightarrow \quad \begin{cases} \operatorname{Im} \beta^* \subset \operatorname{Im} \delta \\ \delta \geq 0 \\ \alpha - \beta\delta^{[-1]}\beta^* \geq 0 \end{cases}$$

Definition: (finite dimensional case)

$H : \mathbb{C}^n \to \mathbb{C}^n$ is a *generalized solution of* ***the Riccati inequality*** associated with Σ if

1. $\langle Hx, x \rangle > 0 \qquad x \neq 0$
2. $(D^*C + B^*HA)\mathbb{C}^n \subset \delta_\Sigma(H)\mathbb{C}^p$
3. $\delta_\Sigma(H) = I - D^*D - B^*HB \geq 0$
4. $H - A^*HA - C^*C - (C^*D + A^*HB)\delta_\Sigma(H)^{[-1]}(D^*C + B^*HA) \geq 0$

$$\begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix} = \begin{bmatrix} H - A^*HA - C^*C & -C^*D - A^*HB \\ -D^*C - B^*HA & I - D^*D - B^*HB \end{bmatrix} \geq 0$$

$$\begin{bmatrix} \alpha & \beta \\ \beta^* & \delta \end{bmatrix} \geq 0 \quad \Leftrightarrow \quad \begin{cases} \operatorname{Im} \beta^* \subset \operatorname{Im} \delta \\ \delta \geq 0 \\ \alpha - \beta\delta^{[-1]}\beta^* \geq 0 \end{cases}$$

Definition: (finite dimensional case)

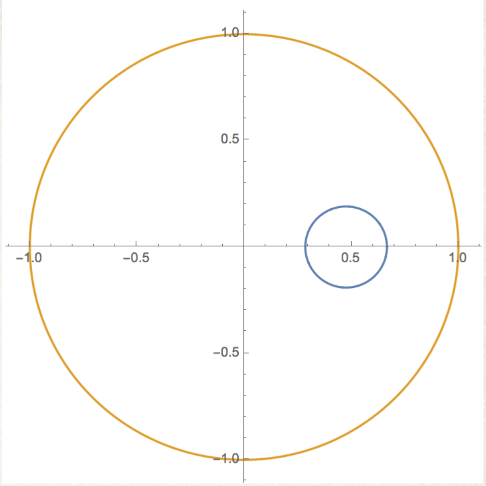
$H : \mathbb{C}^n \to \mathbb{C}^n$ is a *generalized solution of the Riccati equality* associated with Σ if

1. $\langle Hx, x \rangle > 0 \quad x \neq 0$
2. $(D^*C + B^*HA)\mathbb{C}^n \subset \delta_\Sigma(H)\mathbb{C}^p$
3. $\delta_\Sigma(H) = I - D^*D - B^*HB \geq 0$
4. $H - A^*HA - C^*C - (C^*D + A^*HB)\delta_\Sigma(H)^{[-1]}(D^*C + B^*HA) = 0$

Example 1

$$M_\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{c:c} -\frac{1}{8} & 1 \\ \hdashline \frac{3}{16} & \frac{1}{2} \end{array}\right]$$

$$\theta(\lambda) = \frac{2\lambda + 4}{\lambda + 8}.$$



$$|\theta(\lambda)| \leq 6/7 < 1 \text{ for all } \lambda \in \mathbb{D},$$

Notation

$$\begin{aligned} \alpha(H) &= H - A^*HA - C^*C \\ \beta(H) &= C^*D + A^*HB \\ \delta(H) &= I - D^*D - B^*HB \end{aligned}$$

$$R_\Sigma(H) = \alpha(H) - \beta(H)\delta(H)^{[-1]}\beta(H)^*$$

$$\begin{aligned} \alpha(H) &= \frac{9}{64}\left(7H - \frac{1}{4}\right), \\ \beta(H) &= \frac{1}{8}\left(\frac{3}{4} - H\right), \\ \delta(H) &= \frac{3}{4} - H. \end{aligned}$$

Moore Penrose inverse

$$\delta(H)^{[-1]} = \begin{cases} \left(\frac{3}{4} - H\right)^{-1} & (H \neq 3/4) \\ 0 & (H = 3/4) \end{cases} : \mathbb{C} \to \mathbb{C}.$$

Example 1 (continued)

$$M_\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{c:c} -\frac{1}{8} & 1 \\ \hdashline \frac{3}{16} & \frac{1}{2} \end{array}\right] \qquad \theta(\lambda) = \frac{2\lambda+4}{\lambda+8}.$$

$$\begin{aligned} \alpha(H) &= \frac{9}{64}\left(7H - \frac{1}{4}\right), \\ \beta(H) &= \frac{1}{8}\left(\frac{3}{4} - H\right), \\ \delta(H) &= \frac{3}{4} - H. \end{aligned}$$

Moore Penrose inverse

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Riccati function: $R_\Sigma(H) = \alpha(H) - \beta(H)^* \delta(H)^{[-1]} \beta(H)$

1. $\langle Hx, x\rangle > 0 \quad x \neq 0$
2. $\mathrm{Im}\, \beta^* \subset \mathrm{Im}\, \delta$: no conditions on H.
3. The condition $\delta(H) \geq 0$ is the same as $H \leq \frac{3}{4}$

$\Big\}\; 0 < H \leq \frac{3}{4}$

Riccati function:

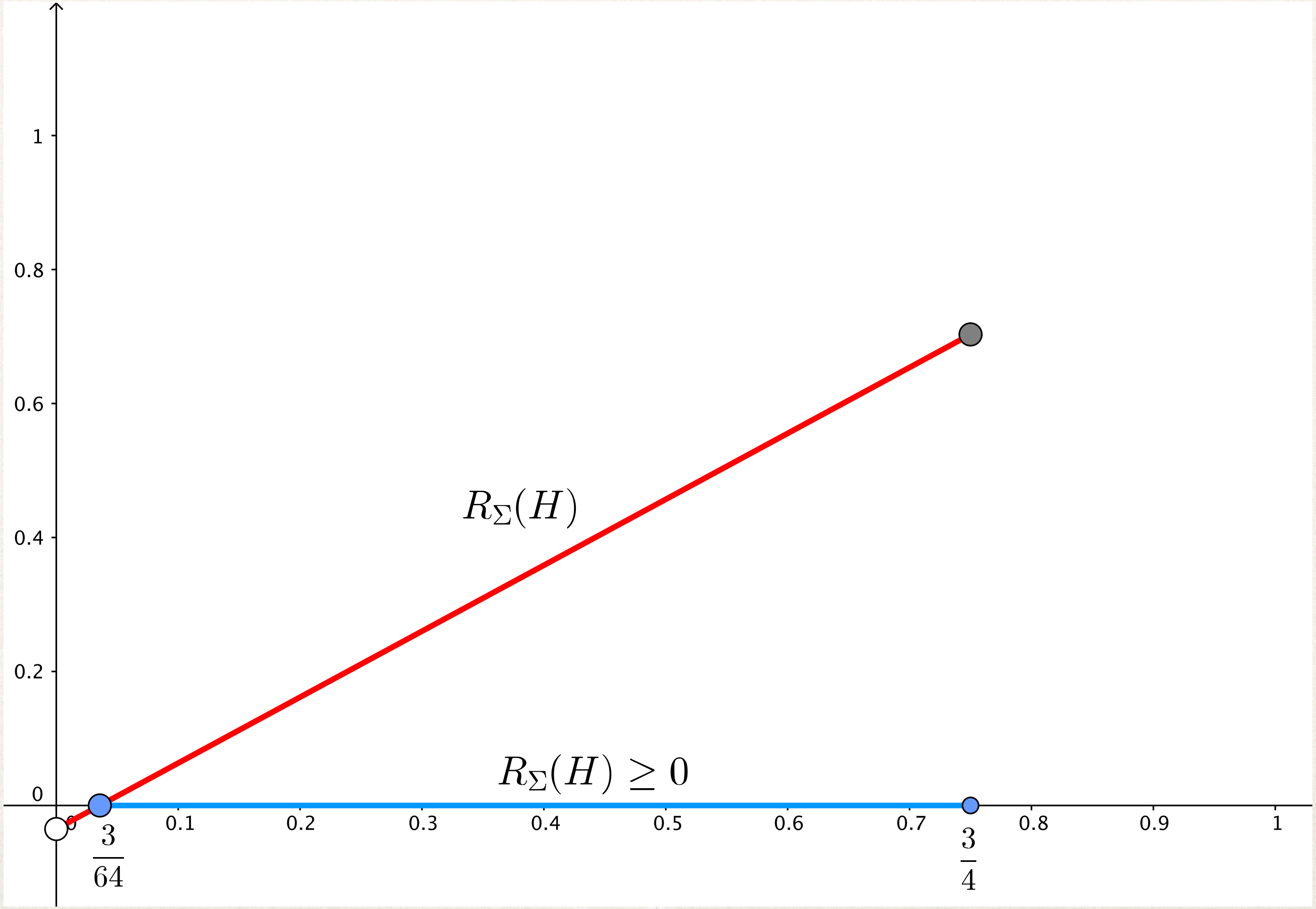
$$R_\Sigma(H) = \begin{cases} H - \frac{3}{64} & (0 < H < 3/4) \\ \frac{9}{64}(7H - \frac{1}{4}) = \frac{45}{64} & (H = 3/4) \end{cases}$$

4. The Riccati equation $R_\Sigma(H) = 0$ has one solution: $H = 3/64$.

$$R_\Sigma(H) = \begin{cases} H - \frac{3}{64} & (0 < H < 3/4) \\ \frac{9}{64}(7H - \frac{1}{4}) = \frac{45}{64} & (H = 3/4) \end{cases}$$

$$R_\Sigma(H) = 0 \qquad H = \frac{3}{64}$$

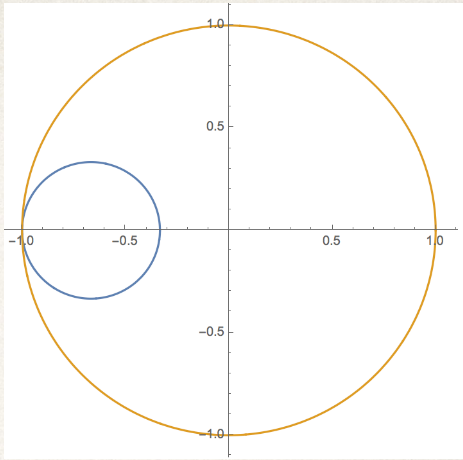
$$R_\Sigma(H) \geq 0 \qquad \frac{3}{64} \leq H \leq \frac{3}{4}$$



Example 2

$$M_\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline \frac{1}{2} & -\frac{5}{6} \end{array}\right]$$

$$\theta_\Sigma(\lambda) = \frac{4\lambda - 5}{3(2-\lambda)}$$



$$|\theta_\Sigma(\lambda)| \le 1 \text{ for all } \lambda \in \mathbb{D}$$

$$\alpha(H) = -\frac{1}{4} + \frac{3}{4}H$$

$$\beta(H) = -\frac{5}{12} + \frac{1}{4}H$$

$$\delta(H) = \frac{11}{36} - \frac{1}{4}H$$

Moore Penrose inverse

$$\delta(H)^{[-1]} = \begin{cases} \left(\frac{11}{36} - \frac{1}{4}H\right)^{-1} & (H \neq \frac{11}{9}) \\ 0 & (H = \frac{11}{9}) \end{cases} : \mathbb{C} \to \mathbb{C}$$

1. $\langle Hx, x\rangle > 0 \quad x \neq 0$
2. $\operatorname{Im} \beta^* \subset \operatorname{Im} \delta$: for $H = \frac{11}{9}$ we have $\beta\left(\frac{11}{9}\right) = -\frac{1}{9}$ and $\delta\left(\frac{11}{9}\right) = 0$ so $H \neq \frac{11}{9}$
3. The condition $\delta(H) \ge 0$ yields $H \le \frac{11}{9}$

$$\left.\right\} \quad 0 < H < \frac{11}{9}$$

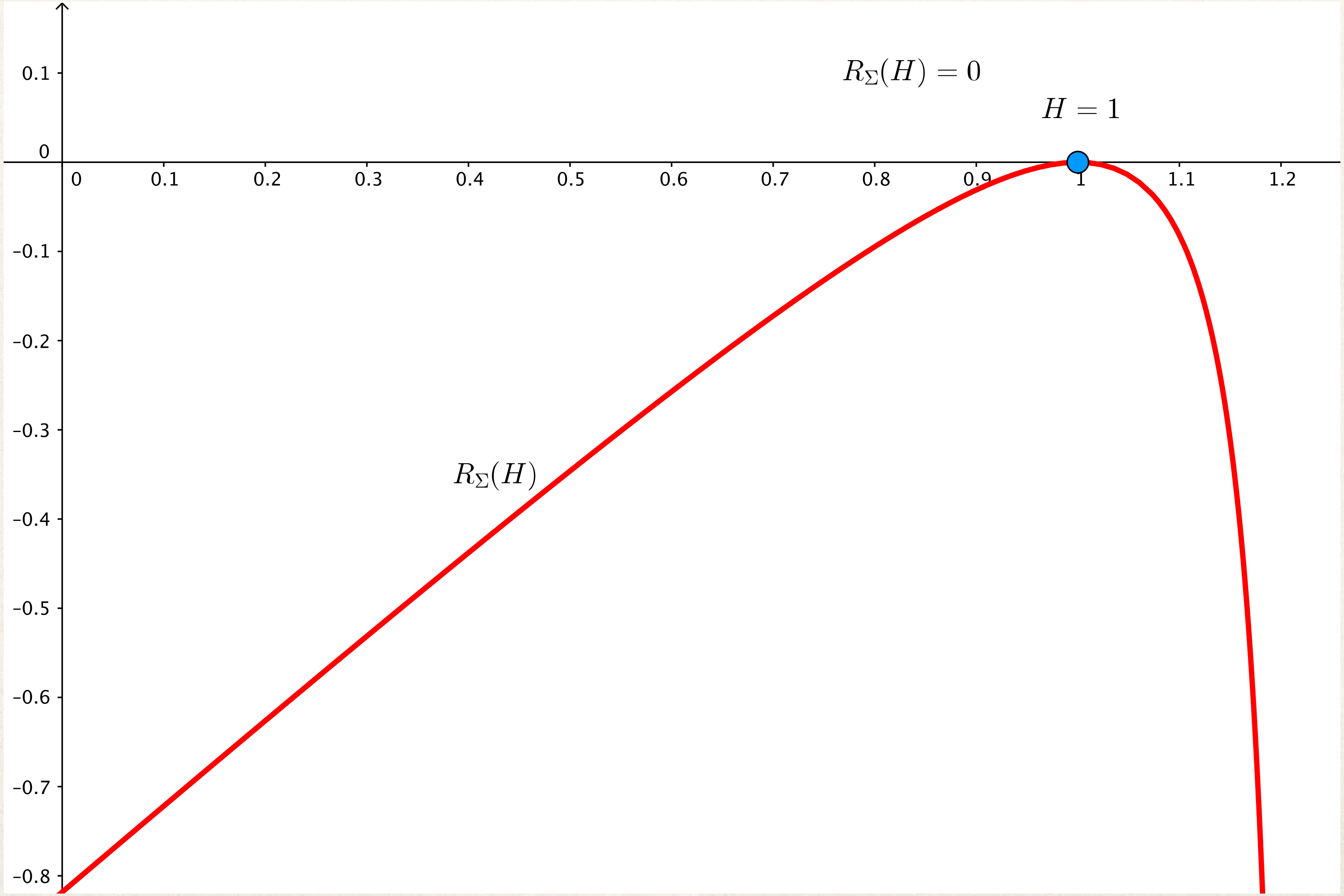
Riccati function:

$$R_\Sigma(H) = \frac{9(H-1)^2}{9H - 11}$$

$$R_\Sigma(H) = \frac{9(H-1)^2}{9H-11}$$

$R_\Sigma(H) = 0$ $\quad H = 1$

$R_\Sigma(H) \geq 0$ $\quad H = 1$



Example 3

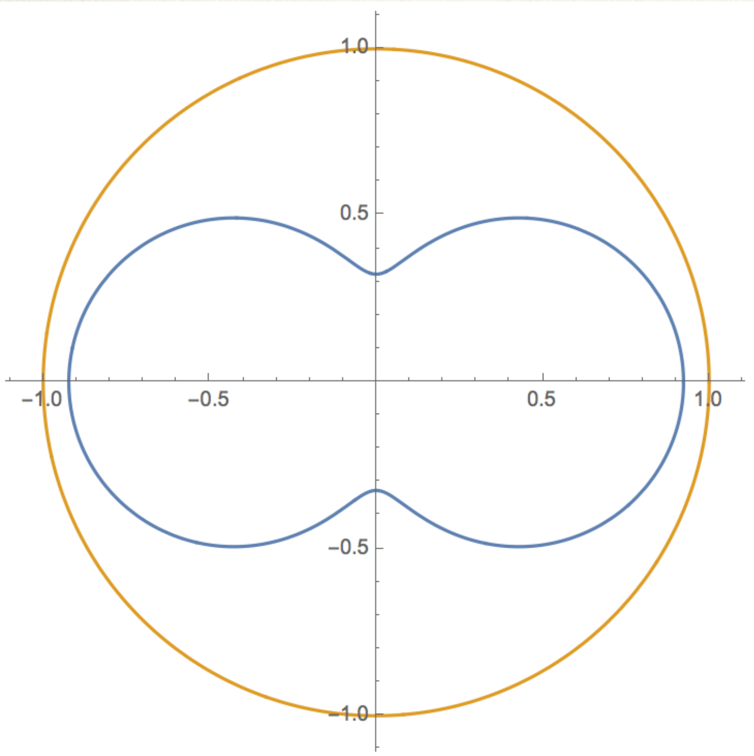
$$\theta(\lambda) = \frac{\lambda ab}{1-\lambda^2 ab}$$

$$0 < a < b < 1, \quad a^2 + b^2 = 1$$

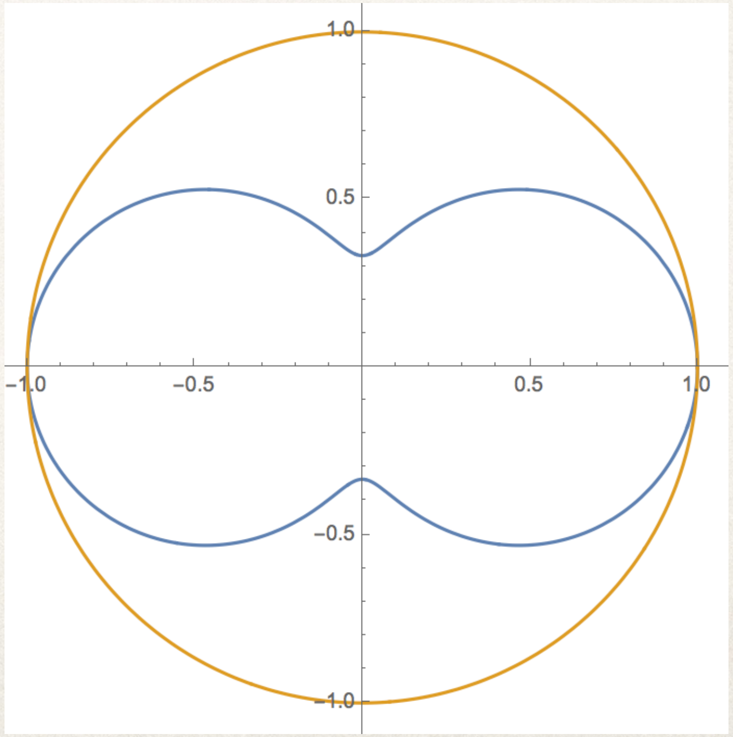
Schur class function

$$M_\Sigma = \left[\begin{array}{cc|c} 0 & a & 0 \\ b & 0 & a \\ \hline 0 & b & 0 \end{array}\right]$$

$$a = \frac{3}{5}, \quad b = \frac{4}{5}$$



$$a = \frac{1}{2}\sqrt{2}, \quad b = \frac{1}{2}\sqrt{2}$$



Example 3, continued

$$\theta(\lambda) = \frac{\lambda ab}{1-\lambda^2 ab}$$ $$0<a<b<1, \quad a^2+b^2=1$$ Schur class function

$$M_\Sigma = \left[\begin{array}{cc|c} 0 & a & 0 \\ b & 0 & a \\ \hline 0 & b & 0 \end{array}\right]$$

$$H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad H_2 = \frac{1}{a^2}\begin{bmatrix} (1-ab)\frac{b}{a} & (b-a)\sqrt{\frac{b}{a}} \\ (b-a)\sqrt{\frac{b}{a}} & 1-ab \end{bmatrix} \quad H_3 = \frac{1}{a^2}\begin{bmatrix} (1-ab)\frac{b}{a} & -(b-a)\sqrt{\frac{b}{a}} \\ -(b-a)\sqrt{\frac{b}{a}} & 1-ab \end{bmatrix}$$

$$H_4 = \frac{1}{a^4}\begin{bmatrix} b^4 & 0 \\ 0 & a^2b^2 \end{bmatrix}$$

$$H_1 \leq H_j \leq H_4 \quad (j=1,2)$$

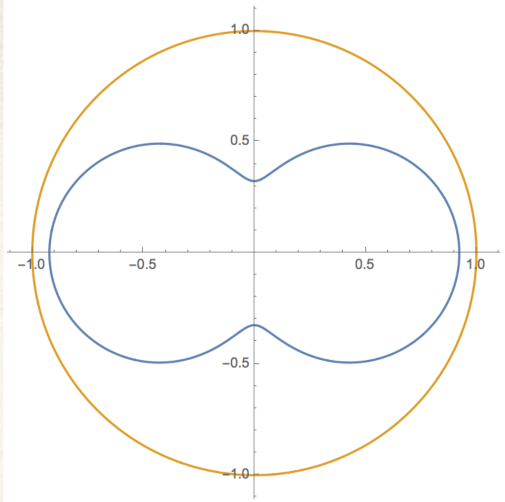
Set of solutions to Riccati equation: $RE_\Sigma = \{H_1, H_2, H_3, H_4\}$

Set of solutions to Riccati inequality, RI_Σ ,
it has minimal element H_1 and maximal element H_4 .

Example 3, continued

$$\theta(\lambda) = \frac{\lambda ab}{1-\lambda^2 ab}$$

$$a = \frac{3}{5}, \quad b = \frac{4}{5}$$



$$M_\Sigma = \left[\begin{array}{cc|c} 0 & a & 0 \\ b & 0 & a \\ \hline 0 & b & 0 \end{array}\right]$$

$$H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H2 = \begin{bmatrix} \frac{52}{27} & \frac{10}{9\sqrt{3}} \\ \frac{10}{9\sqrt{3}} & \frac{13}{9} \end{bmatrix}$$

$$H_3 = \begin{bmatrix} \frac{52}{27} & -\frac{10}{9\sqrt{3}} \\ -\frac{10}{9\sqrt{3}} & \frac{13}{9} \end{bmatrix}$$

$$H_4 = \begin{bmatrix} \frac{256}{81} & 0 \\ 0 & \frac{16}{9} \end{bmatrix}$$

$$H_1 \leq H_j \leq H_4 \quad (j = 1, 2)$$

Set of solutions to Riccati equation: $RE_\Sigma = \{H_1, H_2, H_3, H_4\}$

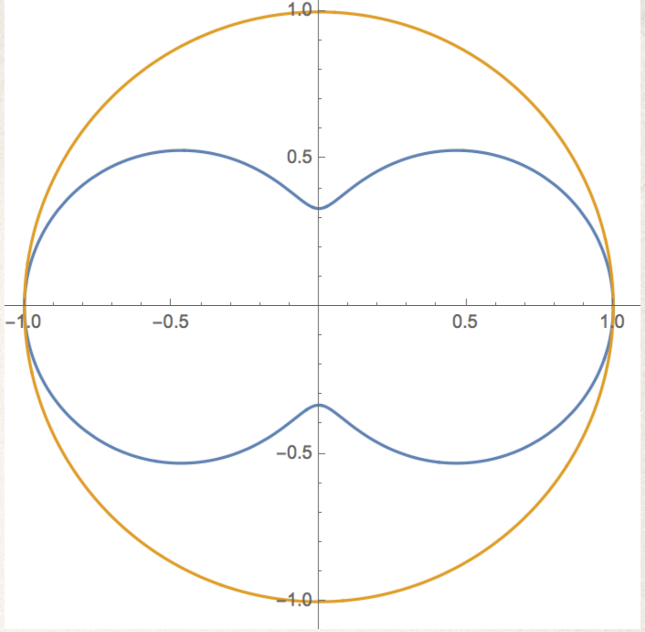
Set of solutions to Riccati inequality, RI_Σ ,
has minimal element H_1 and maximal element H_4 .

Example 3, continued

$$\theta(\lambda) = \frac{\lambda ab}{1 - \lambda^2 ab}$$

$$a = \frac{1}{2}\sqrt{2}, \ b = \frac{1}{2}\sqrt{2}$$

$$M_\Sigma = \left[\begin{array}{cc|c} 0 & a & 0 \\ b & 0 & a \\ \hline 0 & b & 0 \end{array}\right]$$



One single solution $H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to the Riccati equation

and the inequality.

Infinite dimensions: obstacles

- Two minimal systems with same transfer function need not be similar
  So: we use pseudo-similarity (Helton 1974; Arov 1974)

- Minimality not preserved under pseudo-similarity

- For two minimal systems with same transfer function, pseudo-similarity need not be unique

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Definition: (finite dimensional case)

$H : \mathbb{C}^n \to \mathbb{C}^n$ is a *generalized solution of the Riccati equation* associated with Σ if

1. $\langle Hx, x \rangle > 0 \quad x \neq 0$

2. -

3. $\delta_\Sigma(H) = I - D^*D - B^*HB \geq 0$

   and $(D^*C + B^*HA)\mathbb{C}^n \subset \delta_\Sigma(H)\mathbb{C}^p$

4. $H - A^*HA - C^*C - (C^*D + A^*HB)\delta_\Sigma(H)^{[-1]}(D^*C + B^*HA) = 0$

Definition: **(infinite dimensional case)**

$H(\mathcal{X} \to \mathcal{X})$ is a generalized solution of the Riccati equation associated with Σ if

1. $H(\mathcal{X} \to \mathcal{X}), \quad H > 0$ ($\langle Hx, x\rangle > 0 \quad x \neq 0, x \in \mathcal{D}(H)$)

2. $A\,\mathcal{D}(H^{1/2}) \subset \mathcal{D}(H^{1/2}), \quad B\,\mathcal{U} \subset \mathcal{D}(H^{1/2})$

3. $\delta_\Sigma(H) = I_{\mathcal{U}} - D^*D - (H^{1/2}B)^*H^{1/2}B$ is bounded, nonnegative and
$$\left(D^*C + (H^{1/2}B)^*H^{1/2}A\right)\mathcal{D}(H^{1/2}) \subset \delta_\Sigma(H)^{1/2}\mathcal{U}$$

4. $$\forall x \in \mathcal{D}(H^{1/2}) : \quad \|H^{1/2}x\|^2 - \|H^{1/2}Ax\|^2 - \|Cx\|^2 = \|\left(\delta_\Sigma(H)^{1/2}\right)^{[-1]}\left(D^*C + (H^{1/2}B)^*H^{1/2}A\right)x\|^2$$

Moore Penrose Inverse

Bounded, self-adjoint operator $\delta : \mathcal{X} \to \mathcal{X}$

Put $\mathcal{X}_1 = \overline{\delta \mathcal{X}}, \quad \mathcal{X}_2 = \operatorname{Ker} \delta$ and $\delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix}$

Then the Moore Penrose Inverse is defined by

$$\delta^{[-1]} = \begin{bmatrix} \delta_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{Im} \delta_1 \\ \mathcal{X}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix}$$

$$\mathcal{D}(\delta^{[-1]}) = \operatorname{Im} \delta_1 \oplus \mathcal{X}_2$$

Theorem 1

Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system.

If there exists a generalized solution to the Riccati equation associated with Σ,

then the transfer function

$$\theta_\Sigma(\lambda) = D + \lambda C\left(I - \lambda A\right)^{-1} B$$

coincides with a Schur class function in a neighborhood of 0.

Definition: (infinite dimensional case)

$H(\mathcal{X} \to \mathcal{X})$ is a generalized solution of the Riccati equation associated with Σ if

1. $H(\mathcal{X} \to \mathcal{X}), \quad H > 0$ ($\langle Hx, x\rangle > 0 \quad x \neq 0, x \in \mathcal{D}(H)$)
2. $A\,\mathcal{D}(H^{1/2}) \subset \mathcal{D}(H^{1/2}), \quad B\,\mathcal{U} \subset \mathcal{D}(H^{1/2})$
3. $\delta_\Sigma(H) = I_\mathcal{U} - D^*D - (H^{1/2}B)^*H^{1/2}B$ is bounded, nonnegative and $(D^*C + (H^{1/2}B)^*H^{1/2}A)\,\mathcal{D}(H^{1/2}) \subset \delta_\Sigma(H)^{1/2}\mathcal{U}$
4. $\forall x \in \mathcal{D}(H^{1/2}): \quad \|H^{1/2}x\|^2 - \|H^{1/2}Ax\|^2 - \|Cx\|^2 = \|\left(\delta_\Sigma(H)^{1/2}\right)^{[-1]}\left(D^*C + (H^{1/2}B)^*H^{1/2}A\right)x\|^2$

Theorem 2

Let Σ be a minimal system such that its transfer function

$$\theta_\Sigma(\lambda) = D + \lambda C(I - \lambda A)^{-1} B$$

coincides with a Schur class function in a neighborhood of 0.

Then there exists a generalized solution $H(\mathcal{X} \to \mathcal{X})$ to the Riccati equation.

Moreover, the set of all generalized solutions to the Riccati equation has a minimal element.

Definition: (infinite dimensional case)

$H(\mathcal{X} \to \mathcal{X})$ is a generalized solution of the Riccati equation associated with Σ if

1. $H(\mathcal{X} \to \mathcal{X}), \quad H > 0$ ($\langle Hx, x \rangle > 0 \quad x \neq 0, x \in \mathcal{D}(H)$)
2. $A\,\mathcal{D}(H^{1/2}) \subset \mathcal{D}(H^{1/2}), \quad B\,\mathcal{U} \subset \mathcal{D}(H^{1/2})$
3. $\delta_\Sigma(H) = I_{\mathcal{U}} - D^*D - (H^{1/2}B)^* H^{1/2} B$ is bounded, nonnegative and $\left(D^*C + (H^{1/2}B)^* H^{1/2} A\right) \mathcal{D}(H^{1/2}) \subset \delta_\Sigma(H)^{1/2} \mathcal{U}$
4. $\forall x \in \mathcal{D}(H^{1/2}):\ \|H^{1/2}x\|^2 - \|H^{1/2}Ax\|^2 - \|Cx\|^2 = \left\| \left(\delta_\Sigma(H)^{1/2}\right)^{[-1]} \left(D^*C + (H^{1/2}B)^* H^{1/2} A\right) x \right\|^2$

Final remarks

- We have finite and infinite dimensional examples (but we wish for more)
- When does the generalized Riccati equality have one unique solution? (We have theorems in terms of θ)
- What properties do the sets of solutions of the Riccati equality RE_Σ and of the Riccati inequality RI_Σ have? Descriptions in the paper.