Truncated Hankel operators and their matrices

Małgorzata Michalska, Bartosz Łanucha

Maria Curie-Skłodowska University, Lublin, Poland

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Denote:

- H^2 - the Hardy space in the unit disk $\mathbb{D}$,
- P - the orthogonal projection from $L^2 := L^2(\partial\mathbb{D})$ onto H^2,
- $L^\infty := L^\infty(\partial\mathbb{D})$.

For a symbol $\varphi \in L^\infty$ we define:

- T_φ - the Toeplitz operator

$$T_\varphi f = P(\varphi f),$$

- H_φ - the Hankel operator

$$H_\varphi f = J(I - P)(\varphi f),$$

$J : L^2 \to L^2$ is the "flip" operator given by

$$Jf(z) = \overline{z} f(\overline{z}), \quad |z| = 1.$$

We say that α is an inner function if:

- $\alpha \in H^\infty$,
- $|\alpha| = 1$ a.e. on $\partial\mathbb{D}$.

We say that α has an angular derivative in the sense of Carathéodory (ADC) at $w \in \partial\mathbb{D}$, if there exist complex numbers $\alpha(w)$ and $\alpha'(w)$, such that,

$$\alpha(z) \to \alpha(w) \in \partial\mathbb{D} \quad \text{and} \quad \alpha'(z) \to \alpha'(w)$$

whenever $z \to w$ nontangentially (with $\frac{|z-w|}{1-|z|}$ bounded).

Let α be an inner function.

- The corresponding model space K_α is the orthogonal complement of αH^2 in H^2, that is,

$$K_\alpha = H^2 \ominus \alpha H^2.$$

- The model space K_α is a reproducing kernel Hilbert space with the kernel function

$$k_w^\alpha(z) = \frac{1 - \overline{\alpha(w)}\alpha(z)}{1 - \overline{w}z}, \quad w, z \in \mathbb{D}.$$

  (Note that since k_w^α is bounded, the set $K_\alpha^\infty = K_\alpha \cap H^\infty$ is dense in K_α.)
- The conjugate kernel function is the function

$$\widetilde{k}_w^\alpha(z) = \frac{\alpha(z) - \alpha(w)}{z - w}, \quad w, z \in \mathbb{D}.$$

- If α has an ADC at $w \in \partial\mathbb{D}$, then k_w^α and $\widetilde{k}_w^\alpha$, defined as above, belong to K_α.

Truncated Toeplitz and Hankel operators are compressions of classical Toeplitz and Hankel operators to model spaces. More precisely:

- the truncated Toeplitz operator (TTO) A_φ, $\varphi \in L^2$, is densely defined on K_α by

$$A_\varphi f = P_\alpha(\varphi f),$$

  where P_α denotes the orthogonal projection from L^2 onto K_α.
- truncated Hankel operator (THO) B_φ, $\varphi \in L^2$, is densely defined on K_α by

$$B_\varphi = P_\alpha J(I - P)(\varphi f),$$

  where J is the "flip" operator.

For an inner function α let

- $\mathscr{T}(\alpha) = \{A_\varphi : \varphi \in L^2 \text{ and } A_\varphi \text{ is bounded}\}$,
- $\mathscr{H}(\alpha) = \{B_\varphi : \varphi \in L^2 \text{ and } B_\varphi \text{ is bounded}\}$.

Classical Toeplitz and Hankel operators can be characterized in terms of their matrix representations with respect to the monomial basis $\{z^k : k \geq 0\}$ of H^2:

- a bounded linear operator $T : H^2 \to H^2$ is a Toeplitz operator if and only if its matrix is a Toeplitz matrix, that is, it has constant diagonals
- a bounded linear operator $T : H^2 \to H^2$ is a Hankel operator if and only if its matrix is a Hankel matrix, that is, its entries are constant along each skew-diagonal.

The above gives characterizations of TTO and THO in terms of matrix representations with respect to monomial basis of K_α when $\alpha(z) = z^n$.
Matrix representations of operators from $\mathscr{T}(\alpha)$ for other α were discussed by J. A. Cima, W. T. Ross and W. R. Wogen (2008) and by B. Łanucha (2014).

Let α be a finite Blaschke product with distinct zeros $a_1, \dots, a_n$. Then the kernel functions $\{k^\alpha_{a_1}, \dots, k^\alpha_{a_n}\}$ form a basis for K_α and so do the conjugate kernel functions $\{\widetilde{k}^\alpha_{a_1}, \dots, \widetilde{k}^\alpha_{a_n}\}$.
Let B be any linear operator on K_α. Then:

- the matrix representation $M_B = (r_{s,p})$ of B with respect to the kernel basis $\{k^\alpha_{a_1}, \dots, k^\alpha_{a_n}\}$ is given by

$$r_{s,p} = \left(\overline{\alpha'(a_s)}\right)^{-1} \langle B k^\alpha_{a_p}, \widetilde{k}^\alpha_{a_s} \rangle,$$

- the matrix representation $\widetilde{M}_B = (t_{s,p})$ of B with respect to the conjugate kernel basis $\{\widetilde{k}^\alpha_{a_1}, \dots, \widetilde{k}^\alpha_{a_n}\}$ is given by

$$t_{s,p} = \alpha'(a_s)^{-1} \langle B \widetilde{k}^\alpha_{a_p}, k^\alpha_{a_s} \rangle.$$

J. A. Cima, W. T. Ross and W. R. Wogen, 2008

Let α be a finite Blaschke product with n distinct zeros $a_1, \ldots, a_n$. Let A be any linear operator on K_α. If $M_A = (r_{s,p})$ is the matrix representation of A with respect to the basis $\{k^\alpha_{a_1}, \ldots, k^\alpha_{a_n}\}$, then $A \in \mathscr{T}(\alpha)$ if and only if

$$r_{s,p} = \overline{\left(\frac{\alpha'(a_1)}{\alpha'(a_s)}\right)} \left(\frac{r_{1,s}\overline{(a_1 - a_s)} + r_{1,p}\overline{(a_p - a_1)}}{\overline{a_p - a_s}}\right)$$

for all $1 \leq p, s \leq n$, $p \neq s$.

Theorem 1

Let α be a finite Blaschke product with n distinct zeros $a_1, \ldots, a_n$. Let B be any linear operator on K_α. If $M_B = (r_{s,p})$ is the matrix representation of B with respect to the basis $\{k^\alpha_{a_1}, \ldots, k^\alpha_{a_n}\}$, then $B \in \mathscr{H}(\alpha)$ if and only if

$$r_{s,p} = \overline{\left(\frac{1 - a_s a_1}{1 - a_s a_p}\right)} \cdot r_{s,1} - \overline{\left(\frac{\alpha'(a_1)(1 - a_1^2)}{\alpha'(a_s)(1 - a_s a_p)}\right)} \cdot r_{1,1} + \overline{\left(\frac{\alpha'(a_1)(1 - a_1 a_p)}{\alpha'(a_s)(1 - a_s a_p)}\right)} \cdot r_{1,p}$$

for all $1 \leq p, s \leq n$.

C. Gu, Thm. 7.9

Let α be a finite Blaschke product with $n > 0$ zeros.

(a) The dimension of $\mathscr{H}(\alpha)$ is $2n-1$.

(b) If $\lambda_1, \ldots, \lambda_{2n-1}$ are distinct points from $\overline{\mathbb{D}}$, then the operators $k^{\alpha}_{\overline{\lambda_j}} \otimes k^{\alpha}_{\lambda_j}$, $j = 1, \ldots, 2n-1$, form a basis for $\mathscr{H}(\alpha)$.

(c) If $\lambda_1, \ldots, \lambda_{2n-1}$ are distinct points from $\overline{\mathbb{D}}$, then the operators $\widetilde{k}^{\alpha}_{\overline{\lambda_j}} \otimes \widetilde{k}^{\alpha}_{\lambda_j}$, $j = 1, \ldots, 2n-1$, form a basis for $\mathscr{H}(\alpha)$.

Theorem 2

Let α be a finite Blaschke product with n distinct zeros $a_1, \ldots, a_n$. Let B be any linear operator on K_α. If $\widetilde{M}_B = (t_{s,p})$ is the matrix representation of B with respect to the basis $\{\widetilde{k}^\alpha_{a_1}, \ldots, \widetilde{k}^\alpha_{a_n}\}$, then $B \in \mathscr{H}(\alpha)$ if and only if

$$t_{s,p} = \frac{1 - a_s a_1}{1 - a_s a_p} \cdot t_{s,1} - \frac{\alpha'(a_1)(1 - a_1^2)}{\alpha'(a_s)(1 - a_s a_p)} \cdot t_{1,1} + \frac{\alpha'(a_1)(1 - a_1 a_p)}{\alpha'(a_s)(1 - a_s a_p)} \cdot t_{1,p}$$

for all $1 \leq p, s \leq n$.

Let α be an infinite Blaschke product with uniformly separated zeros $\{a_m\}$, that is,

$$\inf_k \prod_{m\neq k}\left|\frac{a_m - a_k}{1-\overline{a_m}a_k}\right| \geq \delta,$$

for some $\delta > 0$. Then the family of kernel functions $\{k^\alpha_{a_m}\}$ forms a basis for K_α and so does the family of conjugate kernel functions $\{\widetilde{k}^\alpha_{a_m}\}$.

This means that each $f \in K_\alpha$ can be written as

$$f = \sum_{m=1}^{\infty} \frac{\overline{\langle f, \widetilde{k}^\alpha_{a_m}\rangle}}{\overline{\alpha'(a_m)}} k^\alpha_{a_m},$$

and

$$f = \sum_{m=1}^{\infty} \frac{\langle f, k^\alpha_{a_m}\rangle}{\alpha'(a_m)} \widetilde{k}^\alpha_{a_m}.$$

Theorem 3

Let α be an infinite Blaschke product with uniformly separated zeros $\{a_m\}$ and let B be a bounded linear operator on K_α. If $M_B = (r_{s,p})$ is the matrix representation of B with respect to the basis $\{k^{\alpha}_{a_m}\}$, then $B \in \mathscr{H}(\alpha)$ if and only if

$$r_{s,p} = \overline{\left(\frac{1 - a_s a_1}{1 - a_s a_p}\right)} \cdot r_{s,1} - \overline{\left(\frac{\alpha'(a_1)(1 - a_1^2)}{\alpha'(a_s)(1 - a_s a_p)}\right)} \cdot r_{1,1} + \overline{\left(\frac{\alpha'(a_1)(1 - a_1 a_p)}{\alpha'(a_s)(1 - a_s a_p)}\right)} \cdot r_{1,p}$$

for all $p, s \geq 1$.

Theorem 4

Let α be an infinite Blaschke product with uniformly separated zeros $\{a_m\}$ and let B be a bounded linear operator on K_α. If $\widetilde{M}_B = (t_{s,p})$ is the matrix representation of B with respect to the basis $\{\widetilde{k}^\alpha_{a_m}\}$, then $B \in \mathscr{H}(\alpha)$ if and only if

$$\begin{aligned} t_{s,p} = &\frac{1 - a_s a_1}{1 - a_s a_p} \cdot t_{s,1} - \frac{\alpha'(a_1)(1 - a_1^2)}{\alpha'(a_s)(1 - a_s a_p)} \cdot t_{1,1} \\ &+ \frac{\alpha'(a_1)(1 - a_1 a_p)}{\alpha'(a_s)(1 - a_s a_p)} \cdot t_{1,p} \end{aligned}$$

for all $p, s \geq 1$.

- For any $\lambda \in \partial\mathbb{D}$ the so-called Clark operator U_λ is the operator from K_α onto K_α defined by

$$U_\lambda = A_z + \frac{\alpha(0)+\lambda}{1-|\alpha(0)|^2} \cdot k_0^\alpha \otimes \widetilde{k}_0^\alpha.$$

- The operator U_λ is unitary and the set of its eigenvalues consists of $\eta \in \partial\mathbb{D}$ such that α has an ADC at η with

$$\alpha(\eta) = \alpha_\lambda = \frac{\alpha(0)+\lambda}{1+\overline{\alpha(0)}\lambda}.$$

  The eigenvector corresponding to η is the reproducing kernel k_η^α (D. N. Clark, 1972).
- If U_λ has a pure point spectrum, then the set $\{v_{\eta_m}\}$, where $\{v_{\eta_m}\} = \{\|k_{\eta_m}^\alpha\|^{-1} k_{\eta_m}^\alpha\}$, is an orthonormal basis for K_α, called the Clark basis corresponding to λ.
- The modified Clark basis is defined by

$$e_{\eta_m} = \omega_m v_{\eta_m}, \quad \text{where} \quad \omega_m = e^{-\frac{i}{2}(\arg\eta_m - \arg\lambda)}.$$

Theorem 5

Let α be a finite Blaschke product with $n > 0$ (not necessarily different) zeros and let $\{v_{\eta_1}, \ldots, v_{\eta_n}\}$ be the Clark basis for K_α corresponding to $\lambda \in \partial\mathbb{D}$. Let B be any linear operator on K_α. If $M_B = (r_{s,p})$ is the matrix representation of B with respect to the basis $\{v_{\eta_1}, \ldots, v_{\eta_n}\}$, then $B \in \mathscr{H}(\alpha)$ if and only if

$$r_{s,p} = \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}} \frac{\eta_s - \overline{\eta_1}}{\eta_s - \overline{\eta_p}} r_{s,1} - \frac{|\alpha'(\eta_1)|}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\alpha'(\eta_s)|}} \frac{\eta_1 - \overline{\eta_1}}{\eta_s - \overline{\eta_p}} r_{1,1} + \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_s)|}} \frac{\eta_1 - \overline{\eta_p}}{\eta_s - \overline{\eta_p}} r_{1,p}$$

for all $1 \leq p, s \leq n$.

Theorem 6

Let α be a finite Blaschke product with $n > 0$ (not necessarily different) zeros and let $\{e_{\eta_1}, \ldots, e_{\eta_n}\}$ be the modified Clark basis for K_α corresponding to $\lambda \in \partial\mathbb{D}$. Let B be any linear operator on K_α. If $\widetilde{M}_B = (t_{s,p})$ is the matrix representation of B with respect to the basis $\{e_{\eta_1}, \ldots, e_{\eta_n}\}$, then $B \in \mathscr{H}(\alpha)$ if and only if

$$t_{s,p} = \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}} \frac{\omega_p}{\omega_1} \frac{\eta_s - \overline{\eta_1}}{\eta_s - \overline{\eta_p}} t_{s,1} - \frac{|\alpha'(\eta_1)|}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\alpha'(\eta_s)|}} \frac{\omega_p}{\omega_s} \frac{\eta_1 - \overline{\eta_1}}{\eta_s - \overline{\eta_p}} t_{1,1} + \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_s)|}} \frac{\omega_1}{\omega_s} \frac{\eta_1 - \overline{\eta_p}}{\eta_s - \overline{\eta_p}} t_{1,p}$$

for all $1 \leq p, s \leq n$.

Remark 1

Theorems 5 and 6 remain true when α is an inner function such that K_α has a Clark basis $\{v_{\eta_m}\}$.

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