Bounded and unbounded solutions to the discrete-time KYP inequality

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Discrete-time linear system and its transfer function

Consider a discrete-time linear system

$$\Sigma := \begin{cases} \mathbf{x}(n+1) &= A\mathbf{x}(n) + B\mathbf{u}(n), \\ \mathbf{y}(n) &= C\mathbf{x}(n) + D\mathbf{u}(n), \end{cases} \qquad (n \in \mathbb{Z})$$

with $\mathbf{u}(n) \in \mathcal{U}$, $\mathbf{x}(n) \in \mathcal{X}$, $\mathbf{y}(n) \in \mathcal{Y}$, where $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$ are Hilbert spaces, and a bounded (linear) system matrix

$$M_\Sigma := \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}.$$

The transfer function of Σ is defined (and analytic) on a neighborhood of 0 by

$$F_\Sigma(z) = D + zC(I - zA)^{-1}B.$$

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Rationale of Bounded Real Lemma

Conditions under which F_Σ has analytic continuation to the unit disk $\mathbb{D}$, also denoted F_Σ, with $\sup_{z\in\mathbb{D}} \|F_\Sigma(z)\| \leq 1$ (standard case) or $\sup_{z\in\mathbb{D}} \|F_\Sigma(z)\| < 1$ (strict case), i.e., F_Σ in $H^\infty_\mathbb{D}(\mathcal{U},\mathcal{Y})$ with $\|F_\Sigma\|_\infty \leq 1$ or $\|F_\Sigma\|_\infty < 1$.

We then say Σ is *(strictly) dissipative*, notation $F_\Sigma \in H^\infty_\mathbb{D}(\mathcal{U},\mathcal{Y})$, $\|F_\Sigma\|_\infty \leq 1$.

Sufficient conditions

State space similarity

Σ is dissipative in case Σ is state space similar to a contractive system:
There exist $\Sigma' = \{A', B', C', D'\}$ and a boundedly invertible $K : \mathcal{X} \to \mathcal{X}'$ with

$$KA = A'K, \quad KB = B', \quad C = C'K, \quad D = D', \quad \left\| \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \right\| \leq 1.$$

Kalman-Yakubovich-Popov (KYP) inequality

Σ is dissipative if there exists a $H > 0$ (positive and bound. invert.) with

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} H & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq \begin{bmatrix} H & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}.$$

Discrete algebraic Riccati form

Taking Schur complement + invertibility assumption, this can be rewritten as:

$$H - A^*HA - C^*C - (A^*HB + C^*D)(I - B^*HB - D^*D)^{-1}(B^*HA + D^*C) \geq 0$$

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Finite dimensional case: $\dim \mathcal{X} < \infty$

Σ minimal, then Σ dissipative iff KYP solution $H > 0$ exists.

Complications in the infinite dimensional case

Complications if $\dim \mathcal{X} = \infty$

- Several notions of minimality, controllable and observable.
- No (direct) generalization of the state space similarity theorem.
- Unbounded solution to KYP-inequality appear, even if M_Σ is bounded.

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Complications if $\dim \mathcal{X} = \infty$

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Various results exist

- Standard case + minimality: No bounded or boundedly invertible KYP sol. H guaranteed. Pseudo solutions [Arov-Kaashoek-Pik '06]. Earlier work: [Arov '74], [Helton '74], [Ball-Cohen '91].
- Standard case + 'exact' minimality: Bounded and boundedly invertible KYP sol. $H > 0$ exists.
- Strict case + $r_{\text{spec}}(A) < 1$: Bounded and boundedly invertible KYP sol. $H > 0$ exists. Implicitly in [Ben-Artzi-Gohberg-Kaashoek '95], variations in [Yakubovich '74, '75].

Willems' storage function approach (1972)

Definition A function $S : \mathcal{X} \to [0, \infty]$ is called a *storage function* if for any system trajectory $(\mathbf{u}(n), \mathbf{x}(n), \mathbf{y}(n))_{n\in\mathbb{Z}}$ we have

$$S(\mathbf{x}(n+1)) \leq S(\mathbf{x}(n)) + \|\mathbf{u}(n)\|^2 - \|\mathbf{y}(n)\|^2 \quad (n \in \mathbb{Z}) \quad \text{and} \quad S(0) = 0.$$

Proposition *Assume the system Σ has a storage function. Then F_Σ has an analytic continuation to $\mathbb{D}$ with $\|F_\Sigma\|_\infty \leq 1$.*

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Available storage, required supply

Assume the system Σ has a storage function. Then we can define storages functions S_a (available storage) and S_r (required supply) by

$$S_a(x_0) = \sup_{n_1 \geq 0} \sum_{n=0}^{n_1} \left(\|\mathbf{y}(n)\|^2 - \|\mathbf{u}(n)\|^2 \right), \quad S_r(x_0) = \inf_{n_{-1} < 0} \sum_{n=n_{-1}}^{-1} \left(\|\mathbf{u}(n)\|^2 - \|\mathbf{y}(n)\|^2 \right)$$

with inf and sup going over all system trajectories satisfying $\mathbf{x}(0) = x_0$, with additional constraint $\mathbf{x}(n_{-1}) = 0$ for the inf. For any storage function S we have

$$S_a(x_0) \leq S(x_0) \leq S_r(x_0) \quad \text{(on a dense domain)}.$$

A storage function S for Σ is called *quadratic* if it has the form

$$S(x) = \langle Hx, x\rangle = \|H^{\frac{1}{2}}x\|^2 \quad (x \in \mathcal{D}(H^{\frac{1}{2}}))$$

where H is a closed, densely defined, injective, positive operator on $\mathcal{X}$ such that

$$A\mathcal{D}(H^{\frac{1}{2}}) \subset \mathcal{D}(H^{\frac{1}{2}}) \quad \text{and} \quad B\mathcal{U} \subset \mathcal{D}(H^{\frac{1}{2}}).$$

Quadratic storage functions and KYP pseudo-solutions

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In that case H satisfied the spatial form of the KYP inequality

$$\left\| \begin{bmatrix} H^{1/2} & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} H^{1/2} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\|^2 \geq 0 \qquad (x \in \mathcal{D}(H^{\frac{1}{2}}),\ u \in \mathcal{U}).$$

Any closed, densely defined, injective, positive operator H on $\mathcal{X}$ satisfying this inequality is called a positive pseudo-solution to the spatial KYP inequality for Σ.

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Conversely, any positive pseudo-solution H to the spatial KYP inequality for Σ provides a quadratic storage function $S_H(x) = \langle Hx, x \rangle$.

The observability and controllability operators

With Σ we associate its observability operator $\mathbf{W}_o : \mathcal{D}(\mathbf{W}_o) \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$ by

$$\mathcal{D}(\mathbf{W}_o) = \{x \in \mathcal{X} \colon \{CA^n x\}_{n\geq 0} \in \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)\}, \quad \mathbf{W}_o x = \{CA^n x\}_{n\geq 0} \ \ (x \in \mathcal{D}(\mathbf{W}_o))$$

and the adjoint controllability operator $\mathbf{W}_c^* : \mathcal{D}(\mathbf{W}_c^*) \to \ell^2_{\mathcal{U}}(\mathbb{Z}_-)$ by

$$\mathcal{D}(\mathbf{W}_c^*) = \{x \in \mathcal{X} \colon \{B^* A^{*(-n-1)} x\}_{n\leq -1} \in \ell^2_{\mathcal{U}}(\mathbb{Z}_-)\}, \quad \mathbf{W}_c^* x = \{B^* A^{*(-n-1)} x\}_{n\leq -1}$$

N.B. It can happen that $\mathcal{D}(\mathbf{W}_o) = \{0\}$ or $\mathcal{D}(\mathbf{W}_c) = \{0\}$ ($C = B = 1$, $A = 2$).

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Proposition *For $\mathbf{W}_o$ and $\mathbf{W}_c^*$ defined above:*

(1) *$\mathbf{W}_o$ is a closed operator on $\mathcal{D}(\mathbf{W}_o)$.*

(2) *Assume that $\mathcal{D}(\mathbf{W}_o)$ is dense in $\mathcal{X}$. Then the adjoint $\mathbf{W}_o^*$ of $\mathbf{W}_o$ exists and is a closed, densely defined operator with domain $\mathcal{D}(\mathbf{W}_o^*)$ containing the linear manifold $\ell_{\text{fin},\mathcal{Y}}(\mathbb{Z}_+)$ of finitely supported sequences in $\ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$.*

(3) *The adjoint controllability operator $\mathbf{W}_c^*$ is closed on $\mathcal{D}(\mathbf{W}_c^*)$.*

(4) *Assume $\mathcal{D}(\mathbf{W}_c^*)$ is dense in $\mathcal{X}$. Then $\mathbf{W}_c^*$ has an adjoint, the controllability operator $\mathbf{W}_c$, which is a closed, densely defined operator with domain $\mathcal{D}(\mathbf{W}_c)$ containing the linear manifold $\ell_{\text{fin},\mathcal{U}}(\mathbb{Z}_-)$ of finitely supported sequences in $\ell^2_{\mathcal{U}}(\mathbb{Z}_-)$.*

Notions of minimality, controllability, observability

Definition Set

$$\text{Rea}\,(A|B) = \text{span}\{\text{Im}\,A^k B\colon\ k = 0,1,2,\ldots\} \quad \text{and} \quad \text{Obs}\,(C|A) = \text{Rea}\,(A^*|C^*).$$

The system Σ (or pair $\{A, B\}$) is called:

- Exactly controllable if $\text{Rea}\,(A|B) = \mathcal{X}$;
- (Approximately) controllable if $\text{Rea}\,(A|B)$ is dense in $\mathcal{X}$.
- ℓ^2-exactly controllable if $\mathcal{D}(\mathbf{W}_c^*)$ is dense in $\mathcal{X}$ and $\text{Im}\,\mathbf{W}_c = \mathcal{X}$.

The system Σ (or pair $\{C, A\}$) is called:

- Exactly observable if $\text{Obs}\,(C|A) = \mathcal{X}$;
- (Approximately) observable if $\text{Obs}\,(C|A)$ is dense in $\mathcal{X}$.
- ℓ^2-exactly observable if $\mathcal{D}(\mathbf{W}_0)$ is dense in $\mathcal{X}$ and $\text{Im}\,\mathbf{W}_o^* = \mathcal{X}$.

We call Σ (exactly/ℓ^2-exactly) minimal if Σ is (exactly/ℓ^2-exactly) controllable and observable.

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- Exactly observable if $\text{Obs}\,(C|A) = \mathcal{X}$;
- (Approximately) observable if $\text{Obs}\,(C|A)$ is dense in $\mathcal{X}$.
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Note: if $\mathcal{D}(\mathbf{W}_c^*)$ and $\mathcal{D}(\mathbf{W}_o)$ are dense in $\mathcal{X}$, then

$$\text{Rea}\,(A|B) = \mathbf{W}_c \ell_{fin,\mathcal{U}}(\mathbb{Z}_-) \quad \text{and} \quad \text{Obs}\,(C|A) = \mathbf{W}_o^* \ell_{fin,\mathcal{U}}(\mathbb{Z}_+).$$

Notions of minimality, controllability, observability II

Proposition

(1) *It can happen that (A, B) is exactly controllable but not ℓ^2-exactly controllable.*

(2) *It can happen that (A, B) is ℓ^2-exactly controllable but not exactly controllable.*

(3) *If (A, B) is exactly controllable, then (A, B) is controllable.*

(4) *If (A, B) is ℓ^2-exactly controllable with $\mathcal{D}(\mathbf{W}_c^*) = \mathcal{X}$, then (A, B) is controllable.*

(5) *If (A, B) is exactly controllable and $\mathcal{D}(\mathbf{W}_c^*)$ is dense, then (A, B) is ℓ^2-exactly controllable.*

(6) *It can happen that (C, A) is exactly observable but not ℓ^2-exactly observable.*

(7) *It can happen that (C, A) is ℓ^2-exactly observable but not exactly observable.*

(8) *If (C, A) is exactly observable, then (C, A) is observable.*

(9) *If (C, A) is ℓ^2-exactly observable and $\mathcal{D}(\mathbf{W}_o) = \mathcal{X}$, then (C, A) is observable.*

(10) *If (C, A) is exactly observable and $\mathcal{D}(\mathbf{W}_o)$ is dense, then (C, A) is ℓ^2-exactly observable.*

Laurent, Toeplitz, Hankel

Assume $F_\Sigma \in H^\infty_{\mathbb{D}}(\mathcal{U},\mathcal{Y})$ with $\|F_\Sigma\|_\infty \leq 1$. Let $(\mathbf{u},\mathbf{x},\mathbf{y})$ be a system trajectory for Σ with $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$. Then

$$\mathbf{y} = L_{F_\Sigma}\mathbf{u}$$

with L_{F_Σ} the Laurent operator defined by F_Σ:

$$L_{F_\Sigma} = \left[\begin{array}{ccc|ccc} \ddots & \ddots & \vdots & \vdots & \vdots & \\ \cdots & F_0 & 0 & 0 & 0 & \cdots \\ \cdots & F_1 & F_0 & 0 & 0 & \cdots \\ \hline \cdots & F_2 & F_1 & F_0 & 0 & \cdots \\ \cdots & F_3 & F_2 & F_1 & F_0 & \ddots \\ & \vdots & \vdots & \vdots & \ddots & \ddots \end{array}\right] = \begin{bmatrix} \widetilde{T}_{F_\Sigma} & 0 \\ H_{F_\Sigma} & T_{F_\Sigma} \end{bmatrix} : \begin{bmatrix} \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \\ \ell^2_{\mathcal{U}}(\mathbb{Z}_+) \end{bmatrix} \to \begin{bmatrix} \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \\ \ell^2_{\mathcal{U}}(\mathbb{Z}_+) \end{bmatrix}$$

and $\|H_{F_\Sigma}\| \leq \|T_{F_\Sigma}\| = \|\widetilde{T}_{F_\Sigma}\| = \|L_{F_\Sigma}\| = \|F_\Sigma\|_\infty \leq 1$.

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and $\|H_{F_\Sigma}\| \leq \|T_{F_\Sigma}\| = \|\widetilde{T}_{F_\Sigma}\| = \|L_{F_\Sigma}\| = \|F_\Sigma\|_\infty \leq 1$. Assume Σ is minimal. Then $\mathcal{D}(\mathbf{W}_o)$ and $\mathcal{D}(\mathbf{W}_c^*)$ are dense and the Hankel operator H_{F_Σ} factors as

$$H_{F_\Sigma}|_{\mathcal{D}(\mathbf{W}_c)} = \mathbf{W}_o\mathbf{W}_c \quad \text{and} \quad H^*_{F_\Sigma}|_{\mathcal{D}(\mathbf{W}_o^*)} = \mathbf{W}_c^*\mathbf{W}_o^*$$

and we have the inclusions

$$\operatorname{Rea}(A|B) \subset \operatorname{Im}\mathbf{W}_c \subset \mathcal{D}(\mathbf{W}_o) \quad \text{and} \quad \operatorname{Obs}(C|A) \subset \operatorname{Im}\mathbf{W}_o^* \subset \mathcal{D}(\mathbf{W}_c^*).$$

Operator forms S_a and S_r

Proposition *Assume* Σ *is minimal with* $F_\Sigma \in H^\infty_{\mathbb{D}}(\mathcal{U}, \mathcal{Y})$ *with* $\|F_\Sigma\|_\infty \le 1$. *Then*

$$S_a(x_0) = \sup_{\mathbf{u}_+ \in \ell^2_{\mathcal{U}}(\mathbb{Z}_+)} \|\mathbf{W}_o x_0 + T_{F_\Sigma} \mathbf{u}_+\|^2_{\ell^2_{\mathcal{Y}}(\mathbb{Z}_+)} - \|\mathbf{u}_+\|^2_{\ell^2_{\mathcal{U}}(\mathbb{Z}_+)},$$

$$S_r(x_0) = \inf_{\mathbf{u}_- \in \ell_{\mathrm{fin},\mathcal{U}}(\mathbb{Z}_-),\, x_0 = \mathbf{W}_o \mathbf{u}_-} \|(I - \widetilde{T}^*_{F_\Sigma} \widetilde{T}_{F_\Sigma})^{\frac{1}{2}} \mathbf{u}_-\|^2.$$

Thus $S_a(x_0) = \infty$ *if* $x_0 \notin \mathcal{D}(\mathbf{W}_o)$ *and* $S_r(x_0) < \infty$ *if and only if* $x_0 \in \mathrm{Rea}\,(A|B)$.

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Thus $S_a(x_0) = \infty$ if $x_0 \notin \mathcal{D}(\mathbf{W}_o)$ and $S_r(x_0) < \infty$ if and only if $x_0 \in$ Rea $(A|B)$.

Proof formula S_a Write P_+ and P_- for the projections on $\ell^2_{\mathcal{Z}}(\mathbb{Z}_+)$ and $\ell^2_{\mathcal{Z}}(\mathbb{Z}_-)$ for any $\mathcal{Z}$. Set $\mathbf{u}_\pm = P_\pm \mathbf{u}$, $\mathbf{y}_\pm = P_\pm \mathbf{y}$. Note $\mathbf{x}(0) = \mathbf{W}_c \mathbf{u}_-$. One can show

$$S_a(x_0) = \sup_{\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z}), x_0 = \mathbf{W}_c \mathbf{u}_-} \|\mathbf{y}_+\|^2 - \|\mathbf{u}_+\|^2.$$

Note

$$\mathbf{y}_+ = P_+ L_{F_\Sigma}\mathbf{u} = H_{F_\Sigma}\mathbf{u}_- + T_{F_\Sigma}\mathbf{u}_+ = \mathbf{W}_o\mathbf{W}_c\mathbf{u}_- + T_{F_\Sigma}\mathbf{u}_+ = \mathbf{W}_o x_0 + T_{F_\Sigma}\mathbf{u}_+,$$

which only relies on $\mathbf{u}_+$. We then find

$$S_a(x_0) = \sup_{\mathbf{u}_+ \in \ell^2_{\mathcal{U}}(\mathbb{Z}_+)} \|\mathbf{W}_o x_0 + T_{F_\Sigma}\mathbf{u}_+\|^2 - \|\mathbf{u}_+\|^2. \quad \square$$

Operator forms S_a and S_r

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Thus $S_a(x_0) = \infty$ if $x_0 \notin \mathcal{D}(\mathbf{W}_o)$ and $S_r(x_0) < \infty$ if and only if $x_0 \in \mathrm{Rea}\,(A|B)$.

Proof formula S_r Similarly as for S_a one finds

$$S_r(x_0) = \inf_{u \in \ell_{fin,\mathcal{U}}(\mathbb{Z}), x_0 = \mathbf{W}_c \mathbf{u}_-} \|\mathbf{u}_-\|^2 - \|\mathbf{y}_-\|^2.$$

In this case

$$\mathbf{y}_- = P_- L_{F_\Sigma}\mathbf{u} = \widetilde{T}_{F_\Sigma}\mathbf{u}_-$$

so that

$$\|\mathbf{u}_-\|^2 - \|\mathbf{y}_-\|^2 = \|\mathbf{u}_-\|^2 - \|\widetilde{T}_{F_\Sigma}\mathbf{u}_-\|^2 = \|(I - \widetilde{T}^*_{F_\Sigma}\widetilde{T}_{F_\Sigma})^{\frac{1}{2}}\mathbf{u}_-\|^2.$$

and the formula for S_r follows. □

Quadratic forms

Question

Are S_a and S_r quadratic storage functions?

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Extended version S_r

To find quadratic storage functions we modify S_r in the following way

$$\widetilde{S}_r(x_0) = \inf_{\mathbf{u}_- \in \mathcal{D}(\mathbf{W}_c),\, x_0 = \mathbf{W}_o \mathbf{u}_-} \|(I - \widetilde{T}^*_{F_\Sigma} \widetilde{T}_{F_\Sigma})^{\frac{1}{2}} \mathbf{u}_- \|^2.$$

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Theorem *Assume Σ is minimal with $F_\Sigma \in H^\infty_{\mathbb{D}}(\mathcal{U}, \mathcal{Y})$ with $\|F_\Sigma\|_\infty \leq 1$. Then there exist closed, densely defined, injective, positive operators H_a and H_r with $\operatorname{Im} \mathbf{W}_c$ contained in their domains, such that*

$$S_a(x_0) = \langle H_a x_0, x_0 \rangle, \quad \widetilde{S}_r(x_0) = \langle H_r x_0, x_0 \rangle \quad (x_0 \in \operatorname{Im} \mathbf{W}_c).$$

Thus H_a and H_r are positive pseudo-solutions to the spatial KYP inequality. Moreover, if $\|F_\Sigma\|_\infty < 1$ and Σ is ℓ^2-exactly minimal, then

$$H_a = \mathbf{W}_o^* (I - T_{F_\Sigma} T_{F_\Sigma}^*)^{-1} \mathbf{W}_o \quad \textit{and} \quad H_r^{-1} = \mathbf{W}_c (I - \widetilde{T}_{F_\Sigma}^* \widetilde{T}_{F_\Sigma})^{-1} \mathbf{W}_c^*$$

and H_a and H_r both bounded and boundedly invertible, and hence strictly positive solutions to the KYP inequality.

Special cases I

Note: if Σ is minimal with $\|F_\Sigma\|_\infty < 1$ and Σ is ℓ^2-exactly minimal, then

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Lemma *Assume F_Σ is in $H^\infty_{\mathbb{D}}(\mathcal{U},\mathcal{Y})$. Then*

- *Σ ℓ^2-exactly controllable $\Rightarrow$ $\mathbf{W}_o$ bounded*
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Lemma *Assume Σ is minimal with F_Σ is in $H^\infty_{\mathbb{D}}(\mathcal{U}, \mathcal{Y})$ with $\|F_\Sigma\|_\infty \leq 1$.*

- *Σ ℓ^2-exactly controllable $\Rightarrow$ H_a bounded*
- *Σ ℓ^2-exactly observable $\Rightarrow$ H_r^{-1} bounded*
- *Σ ℓ^2-exactly minimal $\Rightarrow$ H_a & H_r both bounded and boundedly invertible. In that case all positive pseudo-solutions H to the spatial KYP inequality satisfy $H_a \leq H \leq H_r$. Hence they are in fact bounded, strictly positive solutions to the KYP inequality.*

Special cases II

Regular case: $(I - T_{F_\Sigma} T_{F_\Sigma}^*)$ and $(I - \widetilde{T}_{F_\Sigma}^* \widetilde{T}_{F_\Sigma})$ closed range

In that case generalized inverses $(I - T_{F_\Sigma} T_{F_\Sigma}^*)^+$ and $(I - \widetilde{T}_{F_\Sigma}^* \widetilde{T}_{F_\Sigma})^+$ exist and

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Then:

- $\mathbf{W}_o$ (resp. $\mathbf{W}_c^*$) is bounded if and only if H_a (resp. H_r^{-1}) is bounded.
- $\mathbf{W}_o$ (resp. $\mathbf{W}_c^*$) is bounded below if and only if H_a^{-1} (resp. H_r) is bounded.

The regular case includes $\|F_\Sigma\|_\infty < 1$ but also the case with F_Σ inner.

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The regular case includes $\|F_\Sigma\|_\infty < 1$ but also the case with F_Σ inner.

F_Σ inner

Then $(I - T_{F_\Sigma} T_{F_\Sigma}^*) = P_{\mathrm{Ker}\, T_{F_\Sigma}^*}$ and $(I - \widetilde{T}_{F_\Sigma}^* \widetilde{T}_{F_\Sigma}) = P_{\mathrm{Ker}\, \widetilde{T}_{F_\Sigma}}$ and one obtains

$$H_a = \mathbf{W}_o^* \mathbf{W}_o \quad \text{and} \quad H_r^{-1} = \mathbf{W}_c \mathbf{W}_c^*.$$

$\|F_\Sigma\|_\infty < 1$

Without ℓ^2-minimality there still exists a bounded strictly positive solution H to the KYP solution with $H_a \leq H \leq H_r$, so H_a is bounded and H_r boundedly invertible. However, H_a need not be boundedly invertible and H_r need not be bounded.

Infinite dimensional bounded real lemmas

Theorem (Standard Bounded Real Lemma I)

Assume the system Σ is minimal. Then F_Σ has an analytic continuation to $\mathbb{D}$ with $\|F_\Sigma\|_\infty \leq 1$ if and only if there exists a positive pseudo-solution H of the spatial KYP-inequality defined by Σ.

Theorem (Standard Bounded Real Lemma II)

Assume the system Σ is (ℓ^2-)exactly controllable and (ℓ^2-)exactly observable. Then F_Σ has an analytic continuation to $\mathbb{D}$ with $\|F_\Sigma\|_\infty \leq 1$ if and only if there exists a positive definite solution H of the KYP-inequality defined by Σ. In that case $r_{\text{spec}}(A) \leq 1$ and hence F_Σ is analytic on $\mathbb{D}$.

Theorem (Strict Bounded Real Lemma)

Assume the state operator of Σ satisfies $r_{\text{spec}}(A) < 1$. Then F_Σ is in $H^\infty_{\mathbb{D}}(\mathcal{U}, \mathcal{Y})$ with $\|F_\Sigma\|_\infty < 1$ if and only if there exists a bounded positive definite solution H of the strict KYP-inequality:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} H & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \prec \begin{bmatrix} H & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}.$$

THANK YOU FOR YOUR ATTENTION