Elliptic boundary value problems with complex coefficients and fractional regularity data
(the first order approach)

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The divergence form elliptic equation

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The coefficients \( A(t, x) = A(x) \in L^\infty(\mathbb{R}^n : \mathcal{L}(C^{1+n})) \) are

- \( t \)-independent,
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The coefficients \( A(t, x) = A(x) \in L^\infty(\mathbb{R}^n : L^1(\mathbb{C}^{1+n})) \) are
- \( t \)-independent,
- uniformly elliptic: there exists \( \kappa > 0 \) such that

\[ \text{Re}(A(x)v, v) \geq \kappa |v|^2 \quad \forall v \in \mathbb{C}^{1+n}, \; x \in \mathbb{R}^n, \]
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- not assumed to be real, symmetric, or smooth in any way.
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- not assumed to be real, symmetric, or smooth in any way.
- Maximum principle, existence of fundamental solutions, local Hölder regularity of solutions all fail.
Boundary value problems

For \( \theta \in [-1,0) \) and \( p > 1 \), formulate the Neumann problem

\[
\begin{align*}
\text{div} A \nabla u &= 0 \quad \text{in} \quad \mathbb{R}^{1+n}, \\
||\nabla u||_{\mathcal{T}^p_\theta} &\lesssim ||\partial_\nu A f||_{\dot{\mathcal{H}}^p_\theta}, \\
\lim_{t \to \infty} \nabla \|u(t,\cdot)\| &= 0 \quad \text{in} \quad (\mathcal{S}'/\mathcal{P})_{\mathbb{R}^n}, \\
\lim_{t \to 0} \partial_\nu A u(t,\cdot) &= \partial_\nu A f \in \dot{\mathcal{H}}^p_\theta.
\end{align*}
\]
For $\theta \in [-1,0)$ and $p > 1$, formulate the *Neumann problem* $(N)^p_{\theta,A}$:

$$
(N)^p_{\theta,A} : \left\{ \begin{array}{l}
\text{div} \ A \nabla u = 0 \quad \text{in} \ \mathbb{R}^{1+n}_+,

||\nabla u||_{T^p_{\theta}} \lesssim ||\partial_{\nu A} f||_{\dot{H}^p_{\theta}},

\lim_{t \to \infty} \nabla \|u(t,\cdot) = 0 \quad \text{in} \ (S'/\mathcal{P})(\mathbb{R}^n : \mathbb{C}^n),

\lim_{t \to 0} \partial_{\nu A} u(t,\cdot) = \partial_{\nu A} f \in \dot{H}^p_{\theta}(\mathbb{R}^n : \mathbb{C}).
\end{array} \right.
$$

Say $(N)^p_{\theta,A}$ is well-posed if for every boundary data $\partial_{\nu A} f$ there exists a unique solution $u$ satisfying the given conditions.

Goal: find a useful characterisation of well-posedness of $(N)^p_{\theta,A}$.
Boundary value problems

For $\theta \in [-1, 0)$ and $p > 1$, formulate the *Neumann problem*

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(N)^p_{\theta, A} : \begin{cases}
\text{div } A \nabla u = 0 \quad \text{in } \mathbb{R}^{1+n}_+,

\|\nabla u\|_{T^p_{\theta}} \lesssim \|\partial_{\nu_A} f\|_{\dot{H}^p_{\theta}},

\lim_{t \to \infty} \nabla \| u(t, \cdot) \| = 0 \quad \text{in } (S'/\mathcal{P})(\mathbb{R}^n : \mathbb{C}^n),

\lim_{t \to 0} \partial_{\nu_A} u(t, \cdot) = \partial_{\nu_A} f \in \dot{H}^p_{\theta}(\mathbb{R}^n : \mathbb{C}).
\end{cases}
\]

$T^p_{\theta}$: *weighted tent space* (definition on next slide)
For \( \theta \in [-1, 0) \) and \( p > 1 \), formulate the *Neumann problem* \((N)_{\theta,A}^p\): \[
\begin{align*}
\text{div} \, A \nabla u &= 0 \quad \text{in} \; \mathbb{R}^{1+n}, \\
||\nabla u||_{T^p_{\theta}} &= \lesssim ||\partial_{\nu_A} f||_{\dot{H}^p_{\theta}}, \\
\lim_{t \to \infty} \nabla \|u(t, \cdot)\| &= 0 \quad \text{in} \; (S'/\mathcal{P})(\mathbb{R}^n : \mathbb{C}^n), \\
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(N)_{\theta,A}^p : \begin{cases}
\text{div } A \nabla u = 0 & \text{in } \mathbb{R}^{1+n}, \\
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\lim_{t \to \infty} \nabla \|u(t,\cdot) = 0 & \text{in } (S'/\mathcal{P})(\mathbb{R}^n : \mathbb{C}^n), \\
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tangential gradient: $\nabla_{\|} u = (\partial_1 u, \ldots, \partial_n u)$
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$A$-conormal derivative: $\partial_\nu_A u = e_0 \cdot A \nabla u$ ($e_0$: unit vector in the $t$-direction).
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Say \((N)_{\theta,A}^p\) is well-posed if for every boundary data \( \partial_{\nu_A} f \) there exists a unique solution \( u \) satisfying the given conditions.
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Weighted tent spaces

\[ \|F\|_{T^p_\theta} := \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \int_{B(x,t)} |t^{-\theta} F(t,y)|^2 \frac{dy}{t^{n+1}} \right)^{p/2} dx \right)^{1/p}. \]
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a solution \( u \) to \( (N)^p_{\theta,A} \) with boundary data \( \partial_{\nu_A} f \) must satisfy

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History of tent spaces:

- Unweighted tent spaces (\( \theta = 0 \)): Coifman–Meyer–Stein 1985.
- First definition with \( \theta \neq 0 \): Hofmann–Mayboroda–McIntosh 2011.
- 'General theory': Huang 2016 (complex interpolation and factorisation), A. 2017 (real interpolation and embeddings).

Weighted tent spaces satisfy Hardy–Littlewood–Sobolev-type embeddings:

\[ T^p_0 \theta_0 \ni T^p_1 \theta_1 \quad \left( \frac{p_1}{p} - \frac{p_0}{p} \right) = \frac{n}{\theta_1 - \theta_0}, \quad p_1 \geq p_0. \]
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The first-order approach to elliptic BVPs

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- See boundary data for $(N)^p_{\theta, A}$ as a projection $N_{A, p, \theta}$ of the initial value of a Cauchy–Riemann system.

'Theorem': For a range of parameters $(p, \theta)$ depending on $A$, $(N)^p_{\theta, A}$ is well-posed $\iff N_{A, p, \theta}$ is an isomorphism.
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\[
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\]
Dirac operators and Cauchy–Riemann systems

the Dirac operator $D$ acts on distributions $F : \mathbb{R}^{1+n}_+ \to \mathbb{C}^{1+n}$:

\[
DF = \begin{bmatrix}
0 & \nabla \\
-\nabla & 0
\end{bmatrix} \begin{bmatrix} F_{\perp} \\ F_{\parallel} \end{bmatrix} = \begin{bmatrix}
\text{div} F_{\parallel} \\
-\nabla_{\perp} F_{\parallel} \end{bmatrix}.
\]

Let $B \in L^\infty(\mathbb{R}^n ; L(\mathbb{C}^{1+n}))$ satisfy the same assumptions as $A$ (elliptic, $t$-independent), and define the perturbed Dirac operator $DB$ as an unbounded operator on $L^2(\mathbb{R}^n ; \mathbb{C}^{1+n})$ with natural domain.
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(here $F_\perp : \mathbb{R}^{1+n}_+ \rightarrow \mathbb{C}$ and $F_\parallel : \mathbb{R}^{1+n}_+ \rightarrow \mathbb{C}^n$)
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Let $B \in L^\infty(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^{1+n}))$ satisfy the same assumptions as $A$ (elliptic, $t$-independent), and define the *perturbed Dirac operator* $DB$ as an unbounded operator on $L^2(\mathbb{R}^n : \mathbb{C}^{1+n})$ with natural domain.

The *Cauchy–Riemann system* for $DB$ is

$$\text{(CR)}_{DB} : \begin{cases} \partial_t F + DBF = 0 & \text{in } \mathbb{R}^{1+n}_+ \\ F_{\parallel} \in \mathcal{R}(D), \end{cases}$$

with solutions considered in the usual weak sense.
CR-systems vs. elliptic equations

\[ \nabla_A u := \begin{bmatrix} \partial_{\nu A} u \\ \nabla_\| u \end{bmatrix} \]

A-Conormal gradient: \( \nabla_A u \)
CR-systems vs. elliptic equations

**A-Conormal gradient:** \( \nabla_A u := \begin{bmatrix} \partial_{\nu A} u \\ \nabla_\parallel u \end{bmatrix} \)

This transforms solutions of \( \text{div} \, A \nabla u = 0 \) to solutions of \((\text{CR})_{D\hat{A}}\), where \( \hat{A} \) is a transformed coefficient matrix, and conversely:
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\textbf{Theorem (Auscher–Axelsson–McIntosh 2010)}

\( F \) solves \((\text{CR})_{D\hat{A}} \) \iff \( F = \nabla_A u \) for a (unique) \( u \) such that \( \text{div} \ A \nabla u = 0 \).
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In the splitting \( \mathbb{C}^{1+n} = \mathbb{C} \oplus \mathbb{C}^n \), \( \hat{A} \) is defined by

\[
A =: \begin{bmatrix} A_{\perp \perp} & A_{\perp \|} \\ A_{\| \perp} & A_{\| \|} \end{bmatrix}, \quad \hat{A} := \begin{bmatrix} I & 0 \\ A_{\| \perp} & A_{\| \|} \end{bmatrix}^{-1} \begin{bmatrix} A_{\perp \perp} & A_{\perp \|} \\ 0 & I \end{bmatrix}^{-1}.
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Key steps of the approach:

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DB-adapted Hardy–Sobolev spaces

DB is bisectorial, with bounded $H^\infty$ functional calculus on $\overline{\mathcal{R}(DB)} \subset L^2(\mathbb{R}^n)$ (Axelsson–Keith–McIntosh 2006, Auscher–Axelsson–McIntosh 2010)
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For every $\varphi \in H^\infty(S_\mu)$ we can define $\varphi(DB) \in \mathcal{B}(\overline{\mathcal{R}(DB)})$, and an ‘extension operator’

$$(Q_{\varphi, DB} f)(t, x) = (\varphi(tDB)f)(x) \quad (t > 0, f \in \overline{\mathcal{R}(DB)}).$$
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**DB**-adapted Hardy–Sobolev spaces $H^p_{\theta,DB}$ are formally defined by

$$ \|f\|_{H^p_{\theta,DB}} := \|Q_\varphi, DB f\|_{T^p_\theta}. $$

The norm is (almost) independent of $\varphi$. 
**DB**-adapted Hardy–Sobolev spaces

**DB** is bisectorial, with bounded $H^\infty$ functional calculus on $\overline{\mathcal{R}(DB)} \subset L^2(\mathbb{R}^n)$ (Axelsson–Keith–McIntosh 2006, Auscher–Axelsson–McIntosh 2010)

For every $\varphi \in H^\infty(S_\mu)$ we can define $\varphi(\text{DB}) \in \mathcal{B}(\overline{\mathcal{R}(DB)})$, and an ‘extension operator’

$$(Q_{\varphi, DB} f)(t, x) = (\varphi(t DB) f)(x) \quad (t > 0, f \in \overline{\mathcal{R}(DB)}).$$

**DB**-adapted Hardy–Sobolev spaces $\dot{H}^p_{\theta, DB}$ are **formally** defined by

$$\| f \|_{\dot{H}^p_{\theta, DB}} := \| Q_{\varphi, DB} f \|_{T^p_{\theta}}.$$

The norm is (almost) independent of $\varphi$.

**D**-adapted spaces are ‘classical’:

$$\dot{H}^p_{\theta, D} = \dot{H}^p_{\theta}(\mathbb{R}^n) \oplus (\dot{H}^p_{\theta}(\mathbb{R}^n : \mathbb{C}^n) \cap \mathcal{N}(\text{curl})).$$
Spectral subspaces and Cauchy extensions

Bounded $H^\infty$ calculus of $DB$ on $\overline{\mathcal{R}(DB)}$ extends to adapted spaces $H^p_{\theta, DB}$.
Spectral subspaces and Cauchy extensions

Bounded $H^\infty$ calculus of $DB$ on $\overline{\mathcal{R}(DB)}$ extends to adapted spaces $H^p_{\theta, DB}$.

Useful operators can be constructed:
Spectral subspaces and Cauchy extensions

Bounded $H^\infty$ calculus of $DB$ on $\overline{\mathcal{R}(DB)}$ extends to adapted spaces $H^p_{\theta,DB}$.

Useful operators can be constructed:

- The spectral projections $\chi^+(DB)$ and $\chi^-(DB)$ defined via

$$
\chi^+(z) := 1_{z: \text{Re}(z) > 0}, \quad \chi^-(z) := 1_{z: \text{Re}(z) < 0},
$$

which induce a decomposition $H^p_{\theta,DB} = H^p_{+,\theta,DB} \oplus H^p_{-,\theta,DB}$. The Cauchy extension $C_{DB}f(t) := e^{-tDB}\chi^+(DB)f(t)$ acts as a strongly continuous semigroup on $H^p_{+,\theta,DB}$.
Spectral subspaces and Cauchy extensions

Bounded $H^\infty$ calculus of $DB$ on $\mathcal{R}(DB)$ extends to adapted spaces $H^p_{\theta, DB}$.

Useful operators can be constructed:

- The *spectral projections* $\chi^+(DB)$ and $\chi^-(DB)$ defined via

$$\chi^+(z) := 1_{z: \text{Re}(z)>0}, \quad \chi^-(z) := 1_{z: \text{Re}(z)<0},$$

which induce a decomposition

$$H^p_{\theta, DB} = H^p_{\theta, DB}^+ \oplus H^p_{\theta, DB}^-.$$
Spectral subspaces and Cauchy extensions

Bounded $H^\infty$ calculus of $DB$ on $\mathcal{R}(DB)$ extends to adapted spaces $H_{\theta, DB}^p$.

Useful operators can be constructed:

- The spectral projections $\chi^+(DB)$ and $\chi^-(DB)$ defined via
  $$
  \chi^+(z) := 1_{z: \text{Re}(z)>0}, \quad \chi^-(z) := 1_{z: \text{Re}(z)<0},
  $$
  which induce a decomposition
  $$
  H_{\theta, DB}^p = H_{\theta, DB}^{p,+} \oplus H_{\theta, DB}^{p,-}.
  $$

- The Cauchy extension
  $$
  C_{DB}f(t) := e^{-tDB} \chi^+(DB)f \quad (t > 0)
  $$
  which acts as a strongly continuous semigroup on $H_{\theta, DB}^{p,+}$.
Classification of solutions to $(CR)_{DB}$: endpoint cases

Theorem (Auscher–Mourgoglou 2015, Auscher–Stahlhut 2016)

Fix $p \in (1, \infty)$ such that $H_{-1, DB}^p \simeq H_{-1, D}^p$.

(This identification has a precise, technical interpretation.)

Then $F$ solves $(CR)_{DB}$, $F \in T_{p-1}$ and $\lim_{t \to \infty} F(t) \| = 0$ in $S'/P_{\bot} F = C_{DB} f$ for some (unique) $f \in H^{p+1}_{DB} \subset H_{-1, D}^p$.

In this correspondence, $\| f \|_{H_{-1, D}^p} \simeq \| F \|_{T_{p-1}}$.

$\theta = 0$ case: replace $T_{p-1}$ with a certain nontangential maximal function norm.

Hölder space results ("$\nu$ ≥ $\infty$") are also available.
Classification of solutions to \((CR)_{DB}\): endpoint cases

**Theorem (Auscher–Mourgoglou 2015, Auscher–Stahlhut 2016)**

Fix \( p \in (1, \infty) \) such that \( H^p_{-1, DB} \simeq H^p_{-1, D} \).

(This identification has a precise, technical interpretation.)
Theorem (Auscher–Mourgoglou 2015, Auscher–Stahlhut 2016)

Fix \( p \in (1, \infty) \) such that \( H^p_{-1, DB} \cong H^p_{-1, D} \).
(This identification has a precise, technical interpretation.)

Then

\[
F \text{ solves } (CR)_{DB}, \ F \in T^p_{-1} \text{ and } \lim_{t \to \infty} F(t) = 0 \text{ in } S'/\mathcal{P}
\]
Classification of solutions to \((\text{CR})_{DB}\): endpoint cases

**Theorem (Auscher–Mourgoglou 2015, Auscher–Stahlhut 2016)**

Fix \(p \in (1, \infty)\) such that \(H^p_{-1, DB} \simeq H^p_{-1, D}\).

(This identification has a precise, technical interpretation.)

Then

\[ F \text{ solves } (\text{CR})_{DB}, \ F \in T^p_{-1} \text{ and } \lim_{t \to \infty} F(t)\| = 0 \text{ in } S'/\mathcal{P} \]

\[ \iff \]

\[ F = C_{DB} f \text{ for some (unique) } f \in H^p_{-1, DB} \subset H^p_{-1, D}. \]
Classification of solutions to \((\text{CR})_{DB}\): endpoint cases

Theorem (Auscher–Mourgoglou 2015, Auscher–Stahlhut 2016)

Fix \(p \in (1, \infty)\) such that \(\mathcal{H}^{p}_{-1, DB} \simeq \mathcal{H}^{p}_{-1, D}\).
(This identification has a precise, technical interpretation.)

Then

\[
F \text{ solves } (\text{CR})_{DB}, \ F \in T^{p}_{-1} \text{ and } \lim_{t \to \infty} F(t) = 0 \text{ in } S'/\mathcal{P}
\]

\[
\updownarrow
\]

\[
F = C_{DB} f \text{ for some (unique) } f \in \mathcal{H}^{p,+}_{-1, DB} \subset \mathcal{H}^{p}_{-1, D}.
\]

In this correspondence, \(\|f\|_{\mathcal{H}^{p}_{-1}} \simeq \|F\|_{T^{p}_{-1}}\).
Classification of solutions to $(\text{CR})_{DB}$: endpoint cases

**Theorem (Auscher–Mourgoglou 2015, Auscher–Stahlhut 2016)**

Fix $p \in (1, \infty)$ such that $H^p_{-1, DB} \simeq H^p_{-1, D}$.
(This identification has a precise, technical interpretation.)

Then

$$ F \text{ solves } (\text{CR})_{DB}, \ F \in T^p_{-1} \text{ and } \lim_{t \to \infty} F(t)\| = 0 \text{ in } S'/\mathcal{P} \updownarrow$$

$$ F = C_{DB} f \text{ for some (unique) } f \in H^{p,+}_{-1, DB} \subset H^p_{-1, D}.$$

In this correspondence, $\| f\|^{\dot{H}_{-1}^p} \simeq \| F\|^{T^p_{-1}}$.

$\theta = 0$ case: replace $T^p_{-1}$ with a certain nontangential maximal function norm.
Classification of solutions to \((CR)_{DB}\): endpoint cases

**Theorem (Auscher–Mourgoglou 2015, Auscher–Stahlhut 2016)**

Fix \(p \in (1, \infty)\) such that \(H^p_{-1, DB} \simeq H^p_{-1, D}\).

(This identification has a precise, technical interpretation.)

Then

\[
F \text{ solves } (CR)_{DB}, \quad F \in T^p_{-1} \text{ and } \lim_{t \to \infty} F(t)\|_s = 0 \text{ in } S'/P
\]

\[
\iff
F = C_{DB}f \text{ for some (unique) } f \in H^{p,+}_{-1, DB} \subset H^p_{-1, D}.
\]

In this correspondence, \(\|f\|_{\dot{H}^p_{-1}} \simeq \|F\|_{T^p_{-1}}\).

\(\theta = 0\) case: replace \(T^p_{-1}\) with a certain nontangential maximal function norm.

Hölder space results (‘\(p \geq \infty\)’) are also available.
Theorem (A.–Auscher 2017)

Let $\theta \in (-1, 0)$ and $p \in (1, \infty)$ be such that $H^p_{\theta, DB} \simeq H^p_{\theta, D}$.

This does not follow from the previous theorem by interpolation! Our proof only works for $\theta \in (-1, 0)$. 

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Theorem (A.–Auscher 2017)

Let $\theta \in (-1, 0)$ and $p \in (1, \infty)$ be such that $H^p_{\theta, DB} \simeq H^p_{\theta, D}$. Then

$$F \text{ solves } (CR)_{DB}, \ F \in T^p_{\theta} \text{ and } \lim_{t \to \infty} F(t)_{\|} = 0 \text{ in } S'/\mathcal{P}$$
Classification of solutions to $\text{(CR)}_{DB}$: endpoint cases

**Theorem (A.–Auscher 2017)**

Let $\theta \in (-1, 0)$ and $p \in (1, \infty)$ be such that $H^p_{\theta, DB} \simeq H^p_{\theta, D}$. Then

$$F \text{ solves } \text{(CR)}_{DB}, \ F \in T^p_\theta \text{ and } \lim_{t \to \infty} F(t)_{||} = 0 \text{ in } S'/\mathcal{P}$$

\[ F = C_{DB} f \text{ for some (unique) } f \in H^p_{\theta, DB} \subset H^p_{\theta, D}. \]
Classification of solutions to $(\text{CR})_{DB}$: endpoint cases

**Theorem (A.–Auscher 2017)**

Let $\theta \in (-1, 0)$ and $p \in (1, \infty)$ be such that $H^p_{\theta,DB} \simeq H^p_{\theta,D}$.

Then

$$F \text{ solves } (\text{CR})_{DB}, \ F \in T^p_\theta \text{ and } \lim_{t \to \infty} F(t) = 0 \text{ in } S'/\mathcal{P}$$

$$\iff$$

$$F = C_{DB}f \text{ for some (unique) } f \in H^p_{\theta, DB} \subset H^p_{\theta, D}.$$ 

In this correspondence, $\|f\|_{H^p_\theta} \simeq \|F\|_{T^p_\theta}$. 

This does not follow from the previous theorem by interpolation!

Our proof only works for $\theta \in (-1, 0)$. 

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BVP / $L^\infty$ coefficients / fractional regularity data

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Classification of solutions to \((CR)_{DB}\): endpoint cases

**Theorem (A.–Auscher 2017)**

Let \(\theta \in (-1, 0)\) and \(p \in (1, \infty)\) be such that \(\mathcal{H}^p_{\theta, DB} \simeq \mathcal{H}^p_{\theta, D}\). Then

\[
F \text{ solves } (CR)_{DB}, \quad F \in \mathcal{T}_\theta^p \text{ and } \lim_{t \to \infty} F(t)_{\|} = 0 \text{ in } S'/\mathcal{P}
\]

\[
\Downarrow
\]

\[
F = \mathcal{C}_{DB} f \text{ for some (unique) } f \in \mathcal{H}^p_{\theta, DB} \subset \mathcal{H}^p_{\theta, D}.
\]

*In this correspondence, \(\|f\|_{\mathcal{H}^p_{\theta}} \simeq \|F\|_{\mathcal{T}_\theta^p}\).*

This does *not* follow from the previous theorem by interpolation!

Our proof only works for \(\theta \in (-1, 0)\).
The first-order approach to elliptic BVPs

Approach initiated by Auscher–Axelsson–McIntosh 2010.

Key steps of the approach:

- Identify the second-order equation \( \text{div} \, A \nabla u = 0 \) with a first-order evolution equation (a ‘Cauchy–Riemann system’)
- Classify solutions to Cauchy–Riemann systems in various function spaces (eg. tent spaces) via functional calculus/semigroups
- **See boundary data for \((N)^p_{\theta,A}\) as a projection \(N_{A,p,\theta}\) of the initial value of a Cauchy–Riemann system.**

‘Theorem’: For a range of parameters \((p, \theta)\) depending on \(A\),

\[(N)^p_{\theta,A}\text{ is well-posed} \iff N_{A,p,\theta}\text{ is an isomorphism.}\]
Classification of well-posedness

Suppose $\theta \in [-1, 0]$, $p \in (1, \infty)$, and $H^p_{\theta, D \hat{A}} = H^p_{\theta, D}$.

(required to classify solutions to $(CR)_{D \hat{A}}$ in $T^p_\theta$)
Classification of well-posedness

Suppose $\theta \in [-1, 0]$, $p \in (1, \infty)$, and $\mathbf{H}^p_{\theta, D\hat{A}} = \mathbf{H}^p_{\theta, D}$.

(required to classify solutions to $(CR)_{D\hat{A}}$ in $T^p_\theta$)

$$N_\perp : \mathbf{H}^p_{\theta, D} = \dot{H}^p_{\theta}(\mathbb{R}^n) \oplus (\dot{H}^p_{\theta}(\mathbb{R}^n : \mathbb{C}^n) \cap \mathcal{N}(\text{curl})) \to \dot{H}^p_{\theta}(\mathbb{R}^n).$$
Classification of well-posedness

Suppose \( \theta \in [-1, 0] \), \( p \in (1, \infty) \), and \( \mathbf{H}^p_{\theta,D\hat{A}} = \mathbf{H}^p_{\theta,D} \).

(required to classify solutions to \((\text{CR})_{D\hat{A}}\) in \(T^p_\theta\))

\[
N_\perp : \mathbf{H}^p_{\theta,D} = \dot{\mathbf{H}}^p_{\theta}(\mathbb{R}^n) \oplus (\dot{\mathbf{H}}^p_{\theta}(\mathbb{R}^n : \mathbb{C}^n) \cap \mathcal{N}(\text{curl})) \to \dot{\mathbf{H}}^p_{\theta}(\mathbb{R}^n).
\]

Identify \( \mathbf{H}^{p,+}_{\theta,D\hat{A}} \subset \mathbf{H}^p_{\theta,D} \) and restrict the projection \( N_\perp \) to define

\[
N_{A,p,\theta} : \mathbf{H}^{p,+}_{\theta,D\hat{A}} \to \dot{\mathbf{H}}^p_{\theta}(\mathbb{R}^n).
\]
Classification of well-posedness

Suppose $\theta \in [-1, 0]$, $p \in (1, \infty)$, and $\mathcal{H}^p_{\theta,D\hat{A}} = \mathcal{H}^p_{\theta,D}$.

(required to classify solutions to $(\text{CR})_{D\hat{A}}$ in $T^p_\theta$)

$$N_\perp : \mathcal{H}^p_{\theta,D} = \dot{\mathcal{H}}^p_{\theta}(\mathbb{R}^n) \oplus (\dot{\mathcal{H}}^p_{\theta}(\mathbb{R}^n : \mathbb{C}^n) \cap \mathcal{N}(\text{curl})) \to \dot{\mathcal{H}}^p_{\theta}(\mathbb{R}^n).$$

Identify $\mathcal{H}^{p,+}_{\theta,D\hat{A}} \subset \mathcal{H}^p_{\theta,D}$ and restrict the projection $N_\perp$ to define

$$N_{A,p,\theta} : \mathcal{H}^{p,+}_{\theta,D\hat{A}} \to \dot{\mathcal{H}}^p_{\theta}(\mathbb{R}^n).$$


$$(N)^p_{\theta,A} \text{ is well-posed} \iff N_{A,p,\theta} \text{ is an isomorphism.}$$
Classification of well-posedness

Suppose \( \theta \in [-1, 0] \), \( p \in (1, \infty) \), and \( H^p_{\theta, D \hat{A}} = H^p_{\theta, D} \).

(required to classify solutions to \((\text{CR})_{D \hat{A}}\) in \( T^p_{\theta} \))

\[
N_\perp : H^p_{\theta, D} = \dot{H}^p_{\theta}(\mathbb{R}^n) \oplus (\dot{H}^p_{\theta}(\mathbb{R}^n : \mathbb{C}^n) \cap \mathcal{N}(\text{curl})) \rightarrow \dot{H}^p_{\theta}(\mathbb{R}^n).
\]

Identify \( H^{p, +}_{\theta, D \hat{A}} \subset H^p_{\theta, D} \) and restrict the projection \( N_\perp \) to define

\[
N_{A,p,\theta} : H^{p, +}_{\theta, D \hat{A}} \rightarrow \dot{H}^p_{\theta}(\mathbb{R}^n).
\]


\( (N)^p_{\theta, A} \) is well-posed \( \Leftrightarrow \) \( N_{A,p,\theta} \) is an isomorphism.

Idea: every \( F_0 \in H^{p, +}_{\theta, D \hat{A}} \) is the initial value of a solution \( F = \nabla_A u \) of \((\text{CR})_{D \hat{A}}\),
Classification of well-posedness

Suppose $\theta \in [-1, 0]$, $p \in (1, \infty)$, and $\mathbf{H}^p_{\theta,D\hat{A}} = \mathbf{H}^p_{\theta,D}$. (required to classify solutions to $(\text{CR})_{D\hat{A}}$ in $T^\theta_0$)

$$N_\bot : \mathbf{H}^p_{\theta,D} = \dot{\mathbf{H}}^p_{\theta} (\mathbb{R}^n) \oplus (\dot{\mathbf{H}}^p_{\theta} (\mathbb{R}^n : \mathbb{C}^n) \cap \mathcal{N} (\text{curl})) \to \dot{\mathbf{H}}^p_{\theta} (\mathbb{R}^n).$$

Identify $\mathbf{H}^{p,+}_{\theta,D\hat{A}} \subset \mathbf{H}^p_{\theta,D}$ and restrict the projection $N_\bot$ to define

$$N_{A,p,\theta} : \mathbf{H}^{p,+}_{\theta,D\hat{A}} \to \dot{\mathbf{H}}^p_{\theta} (\mathbb{R}^n).$$


$$(N)^p_{\theta,A} \text{ is well-posed} \iff N_{A,p,\theta} \text{ is an isomorphism}.$$  

Idea: every $F_0 \in \mathbf{H}^{p,+}_{\theta,D\hat{A}}$ is the initial value of a solution $F = \nabla_A u$ of $(\text{CR})_{D\hat{A}}$, and $N_{A,p,\theta} F_0 = (\nabla_A u|_{t=0})_\bot = \partial_{\nu_A} u|_{t=0}$. 

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Consequences

- Duality: well-posedness of $(N)^p_{\theta,A}$ implies well-posedness of $(N)^{p'}_{1-\theta,A^*}$
Consequences

- **Duality:** well-posedness of $(N)_\theta^p, A$ implies well-posedness of $(N)^{-1 - \theta, A}_\theta$.

- **Interpolation of (compatible) well-posedness** (takes too much effort to write rigorously)

Extrapolation: well-posedness of $(N)_\theta^p, A$ implies well-posedness of $(N)^{\tilde{p}}, \tilde{A}$ for $(\tilde{p}, \tilde{\theta})$ near $(p, \theta)$.
Consequences

- Duality: well-posedness of \((N)^p_{\theta,A}\) implies well-posedness of \((N)^{p'-1-\theta}_{A^*}\)

- Interpolation of (compatible) well-posedness (takes too much effort to write rigorously)

- Extrapolation: well-posedness of \((N)^p_{\theta,A}\) implies well-posedness of \((N)^{\tilde{p}}_{\tilde{\theta},A}\) for \((\tilde{p}, \tilde{\theta})\) near \((p, \theta)\)
Consequences

- Duality: well-posedness of \( (N)^p_{\theta,A} \) implies well-posedness of \( (N)^{p'}_{1-\theta,A^*} \)

- Interpolation of (compatible) well-posedness (takes too much effort to write rigorously)

- Extrapolation: well-posedness of \( (N)^p_{\theta,A} \) implies well-posedness of \( (N)^{\tilde{p}}_{\tilde{\theta},A} \) for \( (\tilde{p}, \tilde{\theta}) \) near \( (p, \theta) \)

- Some stability in coefficients: w-p of \( (N)^p_{\theta,A} \) implies w-p of \( (N)^p_{\theta,\tilde{A}} \) for \( \|\tilde{A} - A\|_\infty \) sufficiently small (with some restrictions on \( (p, \theta) \)).
What about Besov spaces?

Replace Hardy–Sobolev spaces ˙\(H^p_\theta\) with Besov spaces ˙\(B^{p,p}_\theta\), and tent spaces ˙\(T^p_\theta\) with \(Z^p_\theta\)-spaces,

\[
||F||_{Z^p_\theta} := \left( \int \int_{\mathbb{R}^n} \left( \int_2^\infty \int_0^t \left| \tau^{\theta}\right|^{\frac{1}{2}} F(\tau,\xi) \right|^2 d\xi d\tau \right)^{p/2} \right)^{1/p}.
\]

Note that

\[
( ˙\(H^p_0\), ˙\(H^p_1\)) \cong ( ˙\(B^{p,p}_\theta\), (classical) \(T^p_0\), \(T^p_1\)) \cong (Z^p_\theta)(\text{A. 2017}).
\]

Our whole theory works identically for \(\theta \in (-1,0)\).

By interpolation, \(w-p\) of \((N^p_0,A)\) and \((N^p_{-1},A)\) implies \(w-p\) of corresponding Neumann problems with boundary data in ˙\(B^{p,p}_\theta\) and gradient in \(Z^{p,p}_\theta\).

Thanks for your attention!

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What about Besov spaces?

Replace Hardy–Sobolev spaces $\dot{H}^p_\theta$ with Besov spaces $\dot{B}^{p,p}_\theta$, and tent spaces $T^p_\theta$ with $Z$-spaces $Z^p_\theta$,

$$||F||_{Z^p_\theta} := \left( \int_{\mathbb{R}^{1+n}} \left( \int_{t/2}^{2t} \int_{B(x,t)} |\tau^{-\theta} F(\tau, \xi)|^2 d\xi d\tau \right)^{p/2} dx \frac{dt}{t} \right)^{1/p}.$$
What about Besov spaces?

Replace Hardy–Sobolev spaces $\dot{H}^p_\theta$ with Besov spaces $\dot{B}^{p,p}_\theta$, and tent spaces $T^p_\theta$ with $Z$-spaces $Z^p_\theta$,

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Note that

$$(\dot{H}^{p_0}_\theta, \dot{H}^{p_1}_\theta)_{\alpha,p} \simeq \dot{B}^{p,p}_\theta \quad \text{(classical)}$$

$$(T^{p_0}_\theta, T^{p_1}_\theta)_{\alpha,p} \simeq Z^p_\theta \quad \text{(A. 2017)}.$$
What about Besov spaces?

Replace Hardy–Sobolev spaces $\dot{H}_\theta^p$ with Besov spaces $\dot{B}_{\theta}^{p,p}$, and tent spaces $T_{\theta}^p$ with $Z$-spaces $Z_{\theta}^p$,

$$||F||_{Z_{\theta}^p} := \left( \int_{\mathbb{R}_+^{1+n}} \left( \int_{t/2}^{2t} \int_{B(x,t)} |\tau^{-\theta} F(\tau, \xi)|^2 \, d\xi \, d\tau \right)^{p/2} \, dx \, dt \right)^{1/p}.$$ 

Note that

$$\left( \dot{H}_{\theta_0}^{p_0}, \dot{H}_{\theta_1}^{p_1} \right)_{\alpha,p} \simeq \dot{B}_{\theta}^{p,p} \quad \text{(classical)}$$

$$\left( T_{\theta_0}^{p_0}, T_{\theta_1}^{p_1} \right)_{\alpha,p} \simeq Z_{\theta}^p \quad \text{(A. 2017)}.$$ 

Our whole theory works identically for $\theta \in (-1,0)$. 

What about Besov spaces?

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$$||F||_{Z_\theta^p} := \left( \int \int_{\mathbb{R}_{+}^{1+n}} \left( \int_{t/2}^{2t} \int_{B(x,t)} |\tau^{-\theta} F(\tau, \xi)|^2 d\xi d\tau \right)^{\frac{p}{2}} dx \frac{dt}{t} \right)^{1/p}.$$

Note that

$$(\dot{H}_\theta^0, \dot{H}_\theta^1)_{\alpha,p} \simeq \dot{B}_\theta^{p,p} \quad \text{(classical)}$$

$$(T_\theta^0, T_\theta^1)_{\alpha,p} \simeq Z_\theta^p \quad \text{(A. 2017)}.$$

Our whole theory works identically for $\theta \in (-1, 0)$.

By interpolation, $w$-$p$ of $\left(N\right)^{p_0}_{0,A}$ and $\left(N\right)^{p_1}_{-1,A}$ implies $w$-$p$ of corresponding Neumann problems with boundary data in $\dot{B}_\theta^{p,p}$ and gradient in $Z_\theta^{p,p}$. 

Thank you for your attention!
What about Besov spaces?

Replace Hardy–Sobolev spaces \( \dot{H}_\theta^p \) with Besov spaces \( \dot{B}_\theta^{p,p} \), and tent spaces \( T_\theta^p \) with \( Z\)-spaces \( Z_\theta^p \),

\[
\| F \|_{Z_\theta^p} := \left( \int_{\mathbb{R}_{+}^{1+n}} \left( \int_{t/2}^{2t} \int_{B(x,t)} |\tau^{-\theta} F(\tau, \xi)|^2 \, d\xi \, d\tau \right)^{p/2} \, dx \, \frac{dt}{t} \right)^{1/p}.
\]

Note that

\[
(\dot{H}_\theta^{p_0}, \dot{H}_\theta^{p_1})_{\alpha,p} \simeq \dot{B}_\theta^{p,p} \quad \text{(classical)}
\]

\[
(T_\theta^{p_0}, T_\theta^{p_1})_{\alpha,p} \simeq Z_\theta^p \quad \text{(A. 2017)}.
\]

Our whole theory works identically for \( \theta \in (-1,0) \).

By interpolation, \( w\)-p of \( (N)_{0,A}^{p_0} \) and \( (N)_{-1,A}^{p_1} \) implies \( w\)-p of corresponding Neumann problems with boundary data in \( \dot{B}_\theta^{p,p} \) and gradient in \( Z_\theta^{p,p} \).

Thanks for your attention!

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