Elliptic boundary value problems with complex coefficients and fractional regularity data

(the first order approach)

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- not assumed to be real, symmetric, or smooth in any way.
- Maximum principle, existence of fundamental solutions, local Hölder regularity of solutions **all fail**.

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Goal: find a useful characterisation of well-posedness of $(N)^p_{\theta,A}$.

Weighted tent spaces

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Weighted tent spaces satisfy Hardy–Littlewood–Sobolev-type embeddings:

$$T^{p_0}_{\theta_0} \hookrightarrow T^{p_1}_{\theta_1} \qquad (p_1 - p_0) = \frac{1}{n}(\theta_1 - \theta_0), \quad p_1 \geq p_0.$$

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'Theorem': For a range of parameters (p, θ) depending on A,

$$(N)^p_{\theta,A} \text{ is well-posed} \Leftrightarrow N_{A,p,\theta} \text{ is an isomorphism.}$$

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(here $F_\perp : \mathbb{R}^{1+n}_+ \to \mathbb{C}$ and $F_\| : \mathbb{R}^{1+n}_+ \to \mathbb{C}^n$)

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The *Cauchy–Riemann system* for DB is

$$(\mathrm{CR})_{DB} : \begin{cases} \partial_t F + DBF = 0 & \text{in } \mathbb{R}^{1+n}_+, \\ F_{\|} \in \overline{\mathcal{R}(D)}, \end{cases}$$

with solutions considered in the usual weak sense.

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In the splitting $\mathbb{C}^{1+n} = \mathbb{C} \oplus \mathbb{C}^n$, $\hat{A}$ is defined by

$$A =: \begin{bmatrix} A_{\perp\perp} & A_{\perp\|} \\ A_{\|\perp} & A_{\|\|} \end{bmatrix}, \quad \hat{A} := \begin{bmatrix} I & 0 \\ A_{\|\perp} & A_{\|\|} \end{bmatrix} \begin{bmatrix} A_{\perp\perp} & A_{\perp\|} \\ 0 & I \end{bmatrix}^{-1}.$$

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For every $\varphi \in H^\infty(S_\mu)$ we can define $\varphi(DB) \in \mathcal{B}(\overline{\mathcal{R}(DB)})$, and an *'extension operator'*

$$(\mathbb{Q}_{\varphi,DB} f)(t,x) = (\varphi(tDB)f)(x) \qquad (t > 0, f \in \overline{\mathcal{R}(DB)}).$$

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$$||f||_{\mathbf{H}^p_{\theta,DB}} := ||\mathbb{Q}_{\varphi,DB} f||_{T^p_\theta} .$$

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D-adapted spaces are 'classical':

$$\mathbf{H}^p_{\theta,D} = \dot{H}^p_\theta(\mathbb{R}^n) \oplus (\dot{H}^p_\theta(\mathbb{R}^n : \mathbb{C}^n) \cap \mathcal{N}(\mathrm{curl}))$$

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- The *Cauchy extension*

  $$\mathbf{C}_{DB} f(t) := e^{-tDB}\chi^+(DB) f \qquad (t > 0)$$

  which acts as a strongly continuous semigroup on $\mathbf{H}^{p,+}_{\theta,DB}$.

Classification of solutions to $(\mathrm{CR})_{DB}$: endpoint cases

Theorem (Auscher–Mourgoglou 2015, Auscher–Stahlhut 2016)

Fix $p \in (1, \infty)$ *such that* $\mathbf{H}^p_{-1,DB} \simeq \mathbf{H}^p_{-1,D}$.

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Hölder space results (‘$p \geq \infty$’) are also available.

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This does *not* follow from the previous theorem by interpolation!

Our proof only works for $\theta \in (-1,0)$.

The first-order approach to elliptic BVPs

Approach initiated by Auscher–Axelsson–McIntosh 2010.

Key steps of the approach:

- Identify the second-order equation $\operatorname{div} A\nabla u = 0$ with a first-order evolution equation (a 'Cauchy–Riemann system')
- Classify solutions to Cauchy–Riemann systems in various function spaces (eg. tent spaces) via functional calculus/semigroups
- **See boundary data for $(N)^p_{\theta,A}$ as a projection $N_{A,p,\theta}$ of the initial value of a Cauchy–Riemann system.**

'Theorem': For a range of parameters (p, θ) depending on A,

$$(N)^p_{\theta,A} \text{ is well-posed} \Leftrightarrow N_{A,p,\theta} \text{ is an isomorphism.}$$

Classification of well-posedness

Suppose $\theta \in [-1, 0]$, $p \in (1, \infty)$, and $\mathbf{H}^p_{\theta, D\hat{A}} = \mathbf{H}^p_{\theta, D}$.

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$$N_\perp : \mathbf{H}^p_{\theta,D} = \dot{H}^p_\theta(\mathbb{R}^n) \oplus (\dot{H}^p_\theta(\mathbb{R}^n : \mathbb{C}^n) \cap \mathcal{N}(\mathrm{curl})) \to \dot{H}^p_\theta(\mathbb{R}^n).$$

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Identify $\mathbf{H}^{p,+}_{\theta, D\hat{A}} \subset \mathbf{H}^p_{\theta, D}$ and restrict the projection $N_\perp$ to define

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Consequences

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- Some stability in coefficients: w-p of $(N)^{p}_{\theta,A}$ implies w-p of $(N)^{p}_{\theta,\tilde{A}}$ for $\|\tilde{A}-A\|_\infty$ sufficiently small (with some restrictions on (p,θ)).

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Replace Hardy–Sobolev spaces $\dot{H}^p_\theta$ with Besov spaces $\dot{B}^{p,p}_\theta$, and tent spaces T^p_θ with Z-*spaces* Z^p_θ,

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Thanks for your attention!