Classification of ill-posedness for bounded linear operators in Banach spaces

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May 20, 2025

Abstract

In this article, concepts of well- and ill-posedness for linear operators in Hilbert and Banach spaces are discussed. While these concepts are well understood in Hilbert spaces, this is not the case in Banach spaces, as there are several competing definitions, related to the occurrence of uncomplemented subspaces. We provide an overview of the various definitions and, based on this, discuss the classification of type I and type II ill-posedness in Banach spaces. Furthermore, a discussion of borderline (hybrid) cases in this classification is given together with several example instances of operators.

MSC2020: 47A52, 47B01, 47B02, 65J20

Keywords: Linear ill-posed operator equation, injective bounded linear operator, ill-posedness types, compact and non-compact operators, strictly singular operators

1 Introduction

Of Hadamard's three requirements for the *well-posedness* of a model problem, namely (i) existence, (ii) uniqueness, and (iii) stability of the solution (cf. [8]), the third demand (iii) is of particular interest for the model of a linear operator equation

$$Ax = y \quad (x \in X,\ y \in Y), \tag{1}$$

formulated in *infinite-dimensional* abstract spaces X and Y (Hilbert or Banach spaces) with some bounded linear forward operator $A : X \to Y$. If at least

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one of the three requirements fails, Hadamard devalues the problems as *ill-posed*. However, for such model problems (1), the requirement (i), which means surjectivity of A, and the requirement (ii), which means injectvity of A, play a less important role for the practical handling, because exact data y belonging to the range $\mathcal{R}(A)$ of A ensure solution existence, and in case of non-trivial null-spaces $\mathcal{N}(A)$ of A uniquely determined elements like minimum-norm solutions lead to the required uniqueness. Moreover, developed mathematical techniques cover the case of multiple solutions. On the other hand, the stability requirement (iii) is indispensable for well-posedness, and its failure cannot be overcome by simple technical tricks.

Here, instability in the context of the violation of Hadamard's requirement (iii) means that small changes in the data (right-hand side of equation (1)) can lead to arbitrarily large changes in the solution. For a stability concept in the case of multiple solutions, we refer to Remark 1 in Section 5 below. For linear operator equations, stability and instability can be expressed by the fact that the inverses $A^{-1} : \mathcal{R}(A) \subset Y \to X$ for injective operators A (and their generalizations for non-injective A—see Section 4) are *bounded* and *unbounded* linear operators, respectively.

The stability aspect has been the substantial reason for M. Z. Nashed (cf. [17]) to characterize well-posedness and ill-posedness of linear operator equations with focus only on the stability requirement (iii). Justified by regularizing properties, Nashed has furthermore distinguished in [17] the type I of ill-posedness from the type II of ill-posedness. First we present a simplified (naive) version of Nashed's definition as Definition 1, which is completely satisfactory for X and Y infinite-dimensional *Hilbert spaces* as well as for *injective* bounded linear operators A when X or Y or both are infinite-dimensional Banach spaces. Definition 3 in Section 4 presents a refinement to do justice to the general *Banach space*, where the null-space $\mathcal{N}(A)$ is not necessarily *complemented* in the Banach space X. This may occur when X is not isomorphic to a Hilbert space. Note that we will write *$\mathcal{R}$-well-posed* and *$\mathcal{R}$-ill-posed* if we refer to Definition 1 in order to distinguish it from *well-posed* and *ill-posed* of the concept by Definition 3 below.

Definition 1 (Naive well-posedness and ill-posedness characterization). *Let $A : X \to Y$ be a bounded linear operator mapping between the infinite-dimensional Banach spaces X and Y.*

Then the operator equation (1) *is called* well-posed *(or* $\mathcal{R}$-well-posed*)*

$$\textit{if the range } \mathcal{R}(A) \textit{ of } A \textit{ is a closed subset of } Y.$$

Consequently (1) *is called ill-posed (or* $\mathcal{R}$-ill-posed*)*

$$\textit{if the range } \mathcal{R}(A) \textit{ is not closed, i.e., } \mathcal{R}(A) \neq \overline{\mathcal{R}(A)}^{Y}.$$

In the ill-posed case, the equation (1) is called ill-posed of type I *(or* $\mathcal{R}$-ill-posed of type I*)*

$$\textit{if the range } \mathcal{R}(A) \textit{ contains an infinite-dimensional closed subspace,}$$

and it is called ill-posed of type II (or $\mathcal{R}$-ill-posed of type II*) otherwise.*

For X and Y being both *Hilbert spaces*, we illustrate by Figure 1 in Section 2, which is analog to [13, Fig. 1], this case distinction with respect to well-posedness and the different types of ill-posedness for bounded linear operators A in the Hilbert space setting. As can be seen in the sequel, the situation in the Banach space setting is a bit more complex, because in Banach spaces non-compact operators can occur that also lead to ill-posendness of type II. In Section 3, we will explain this in detail by Figure 2, which we recall from [5]. Both figures correspond to Definition 1. Illustrated by Figure 3, Section 4 outlines the general Banach space situation, where uncomplemented null-spaces may occur that find its place in the Definition 3 along the lines of Nashed's seminal publication [17] for the characterization of well-posedness and ill-posedness in the general case.

We precede the concept of *strictly singular* operators by Definition 2, which is needed in the following sections, and we also give in this context a list of well-known assertions that characterize strict singularity.

Definition 2 (Strictly singular operator (cf. [7]))**.** *Let X and Y be Banach spaces, and let A be a bounded linear operator mapping X into Y. Then A is said to be* strictly singular *if, given any infinite-dimensional subspace M of X, A restricted to M is not an isomorphism (i.e., linear homeomorphism).*

Proposition 1 ([12])**.** *A bounded linear operator $A : X \to Y$ mapping between Banach spaces is strictly singular if the closed subspaces Z of X, for which the restriction $A|_Z$ has a bounded inverse, are necessarily finite dimensional.*

Evidently, all compact operators are strictly singular. In Hilbert spaces, the converse is true as well:

Proposition 2 ([12, Remark p. 287f])**.** *Let $A : X \to Y$ be a bounded operator mapping between the Hilbert spaces X and Y. Then A is strictly singular if and only if A is compact.*

Lemma 1. *If the range $\mathcal{R}(A)$ of the bounded linear operator $A : X \to Y$ mapping between infinite-dimensional Banach spaces is finite dimensional, then A is strictly singular and compact with closed range.*

Conversely, let A be strictly singular and possess a closed range $\mathcal{R}(A)$. Then $\mathcal{R}(A)$ is finite dimensional and A is also compact whenever either

- *X and Y are Hilbert spaces*
- *or A is an injective map between Banach spaces X and Y.*

Proof. The first assertion follows by the definition of strictly singular. For the second assertion, consider a strictly singular operator with closed range. In both cases, if X, Y are Hilbert spaces or A is injective we may find a continuous right inverse on $\mathcal{R}(A)$: In the Hilbert space case, the pseudo-inverse $A^\dagger$ and in the injective Banach space case, this follows from the open mapping theorem. A classical result of Kato [12] states that the strictly singular operators form an ideal. Hence the composition of A with one of the above inverses gives the identity on $\tilde{X} = \mathcal{N}(A)^\perp$ in the Hilbert case or $\tilde{X} = X$ in the injective Banach space case, which must be strictly singular then. It is well-known [12] that this can only be the case if $\tilde{X}$ is finite dimensional, which implies that the range is so as well. □

Proposition 3 ([7])**.** *Every bounded linear operator* $A : \ell^p \to \ell^q$ *with* $p, q \in [1, \infty)$ *and* $p \neq q$ *is strictly singular.*

In particular, any *embedding operator* $\ell^p \to \ell^q$ with $p < q$ is strictly singular and moreover *non-compact.*

For the next two propositions, see [14, Proposition 2.C.3].

Proposition 4 (Pitt's theorem)**.** *Let* $p, r \in [1, \infty)$ *with* $r > p$*. Then every bounded linear operator from* $\ell^r \to \ell^p$ *is compact. Also any bounded linear map* $A : c_0 \to \ell^p$ *with* $p \in [1, \infty)$ *is compact.*

Proposition 5. *Let* $p \in [1, \infty)$*. Then a bounded linear map* $\ell^p \to \ell^p$ *is strictly singular if and only if it is compact.*

The next result from [7, Corollary III.3.6] extends some of the previous propositions.

Proposition 6. *Let* X *be* $C(K)$ *with* $K \subset \mathbb{R}^n$ *compact, or* ℓ^1*,* ℓ^∞*,* $L^1(\Omega)$*, or* $L^\infty(\Omega)$ *or the dual of any of these spaces. Then any bounded linear map* $X \to Y$ *with* Y *a reflexive Banach space is strictly singular. In particular, this holds for all bounded maps* $L^1(\Omega) \to L^p(\Omega)$ *or* $L^\infty(\Omega) \to L^p(\Omega)$ *with* $p \in (1, \infty)$*.*

The sections of this work are organized as follows. In Section 2, the Hilbert space setting is discussed, and the specific situation there is illustrated by Figure 1. The Banach space setting for injective bounded linear operators will be outlined in Section 3 with respect to well- and ill-posedness, where now non-compact operators appear that are ill-posed of type II in the sense of Definition 1. The situation of this section is illustrated by Figure 2. Section 4 plays a central role for this paper, because there for the general case of bounded linear operators in Banach spaces, an adapted Definition 3 is introduced and illustrated by Figure 3. This definition covering complemented and uncomplemented null-spaces highlights well-posedness and ill-posedness of type I and type II in the sense of Nashed with the occurring distinguished facets of this Banach space world. Section 5 completes the paper by presenting and evaluating a series of alternative concepts of well-posedness from the literature. In Subsection 5.1, characterizations of well-posedness by generalized inverses are given, whereas in Subsection 5.2 alternative concepts for types of ill-posedness are under consideration, illustrated by Figure 4 and a couple of enlightening examples in Section 5.

2 Hilbert space setting

The diagram in Figure 1 below illustrates, under the auspices of Definition 1, which means that well-posed and ill-posed is to understand in the sense of $\mathcal{R}$-well-posed and $\mathcal{R}$-ill-posed, the different cases that are possible for equation (1)

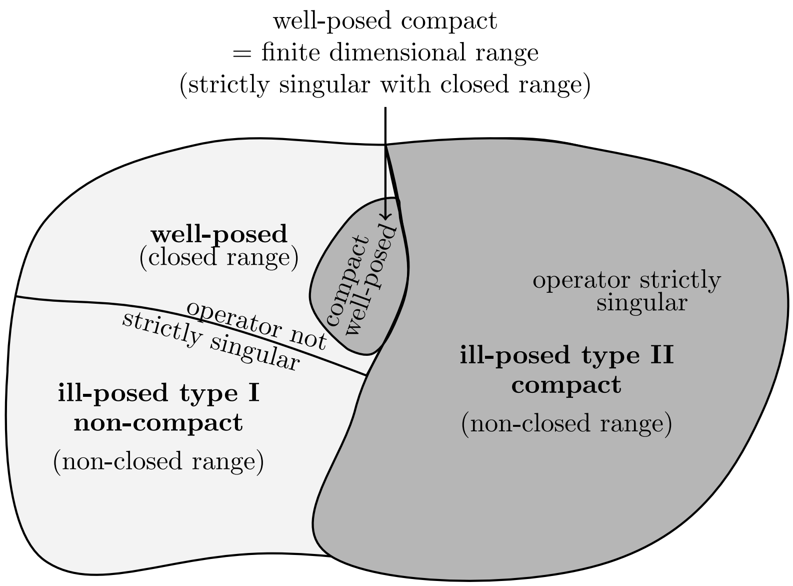

Figure 1: Case distinction for bounded linear operators between infinite-dimensional *Hilbert spaces*. Here, strictly singular operators are always compact.

in the Hilbert space setting, where the bounded linear operator $A : X \to Y$ maps between the infinite-dimensional Hilbert spaces X and Y.

Compact and thus strictly singular operators with infinite-dimensional range possess a non-closed range and are therefore $\mathcal{R}$-ill-posed. This class of operators, which are characterized by an $\mathcal{R}$-ill-posedness of type II, fills out the right part of Figure 1. Compact operators with finite-dimensional range are also strictly singular, but $\mathcal{R}$-well-posed and belong with its closed range (cf. Lemma 1) to the left part of Figure 1. The main part of the left half of this figure consists of not strictly singular operators, the range of which posssesses a closed infinite-dimensional subspace of Y. These operators are either $\mathcal{R}$-well-posed with closed range or alternatively $\mathcal{R}$-ill-posed of type I with non-closed range. $\mathcal{R}$-well-posedness takes place there for continuously invertible operators with the identity operator as the simplest example. $\mathcal{R}$-ill-posed operators of type I can, for example, be found in the class of bounded multiplication operators with essential zeros of the multiplier functions (cf. [15, 19]) and by Cesaro and Hausdorff moment operators (cf. [6, 13]).

Let us emphasize that in Hilbert spaces under Definition 1 ill-posed situations are clearly distinguished by the fact that all ill-posed operators of type II are compact, whereas all ill-posed operators of type I are non-compact. The next section will show that this is not in general true in Banach spaces.

3 Banach space setting for injective operators

The diagram in Figure 2 illustrates, under the auspices of Definition 1, the different cases occurring for equation (1) in the Banach space setting when the bounded linear operator $A : X \to Y$ mapping between the infinite-dimensional Banach spaces X and Y is *injective*. It follows from Definition 2 that strictly singular operators A cannot lead to $\mathcal{R}$-well-posedness or $\mathcal{R}$-ill-posendess of type I for the operator equation (1), because for injective strictly singular operators the range $\mathcal{R}(A)$ of A does not contain an infinite-dimensional closed subspace of Y. Vice versa, "any bounded linear operator between Banach spaces whose range does not contain any infinite-dimensional closed subspace is strictly singular" ([7, Remark]). Consequently, the property of strict singularity separates the left part of Figure 2 from the right part.

Note that here the area of compact operators is a proper subset of the area of strictly singular operators (right part of Figure 2) that are always $\mathcal{R}$-ill-posed of type II (see, e.g., [5, Prop. 4.6]). On the one hand, for such injective bounded linear operators A, finite-dimensional ranges $\mathcal{R}(A)$ do not occur. On the other hand, in Banach spaces, injective bounded non-compact and strictly singular linear operators exist (see Example 2 below concerning embedding operators $\mathcal{E}$ from ℓ^p to ℓ^q for $p < q$), which proves that the compact operators are not enough to fill out the right part of the figure. This indicates a substantial difference between Figure 1 and Figure 2, because compact operators with infinite-dimensional range completely fill out the right part of Figure 1 in the Hilbert space setting.

Examples for compact operators in Banach spaces can be found by injective bounded operators $\mathcal{E} : \ell^p \to \ell^r$ with $r < p$ and $1 \le p, r < \infty$ due to Pitt's theorem (cf. Proposition 4). Also injective linear Fredholm integral operators mapping in $C[0,1]$ with continuous non-degenerating kernel functions represent examples of compact operators with infinite-dimensional range.

Not strictly singular injective operators A (left part of Figure 2), the range of which possesses an infinite-dimensional closed subspace of Y, lead either to $\mathcal{R}$-*well-posed* operator equations (1) with closed range if A is *continuously invertible* or to equations $\mathcal{R}$-*ill-posed of type I* if A has a non-closed range $\mathcal{R}(A) \neq \overline{\mathcal{R}(A)}^Y$.

4 Banach space setting for general operators

The characterization of well-posedness and ill-posedness of operator equations (1) with *not necessarily injective* bounded linear operators $A : X \to Y$ mapping between infinite-dimensional Banach spaces X and Y in Nashed's sense has to take into account that the null-space $\mathcal{N}(A)$ of the operator A need not be (topologically) complemented in X. By considering the direct sum $X = \mathcal{N}(A) \oplus U$, the null-space $\mathcal{N}(A)$ is called *complemented* in X if U is also a closed subspace of X as the subspace $\mathcal{N}(A)$ is, otherwise $\mathcal{N}(A)$ is called *uncomplemented* in X. Banach spaces X, which are not isomorphic to a Hilbert space, always contain uncomplemented subspaces (see, e.g., [14]). Consequently, for the general Ba-

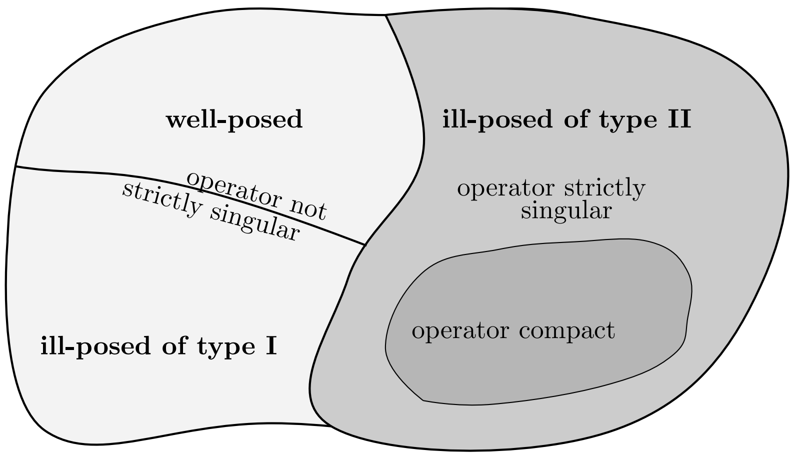

Figure 2: Case distinction for *injective* bounded linear operators mapping between infinite-dimensional *Banach spaces* (cf. [5, p.287]).

nach space setting, Definition 1 has to be replaced by the following Definition 3 which covers both the case of complemented null-space $\mathcal{N}(A)$ and the case that $\mathcal{N}(A)$ is uncomplemented in X.

Definition 3 (Well- and ill-posedness characterization)**.** *Let $A : X \to Y$ be a bounded linear operator mapping between the infinite-dimensional Banach spaces X and Y.*

Then the operator equation (1) *is called* well-posed *if*

> *the range $\mathcal{R}(A)$ of A is a closed subset of Y and, moreover,*
> *the null-space $\mathcal{N}(A)$ is* complemented *in X;*

otherwise the equation (1) is called ill-posed.

In the ill-posed case, (1) is called ill-posed of type I *if*

> *the range $\mathcal{R}(A)$ contains an* infinite-dimensional closed subspace *M of Y and the null-space $\mathcal{N}(A)$ is* complemented *in $A^{-1}[M]$.*

Here, $A^{-1}[M]$ is the inverse image of M under A.

Otherwise the ill-posed equation (1) is called ill-posed of type II.

The following note is a consequence of the corresponding discussions in [4, Section 1]: For the linear bounded operator $A : X \to Y$ defined on the Banach space X with a direct sum decomposition as $X = \mathcal{N}(A) \oplus U$, the restriction $A|_U : U \to \mathcal{R}(A)$ of A is a bijective mapping, and we denote its inverse $A_U^\dagger : \mathcal{R}(A) \to U$ as *generalized pseudoinverse* of A. In the injective case, $A_U^\dagger$ and $A^{-1} : \mathcal{R}(A) \to X$ coincide.

Proposition 7 ([4, Proof of Prop. 1.10 and Prop. 1.11])**.** *If, for $A : X \to Y$ mapping between infinite-dimensional Banach spaces X and Y, the null-space $\mathcal{N}(A)$ is complemented in X, then the generalized pseudoinverse $A_U^\dagger$ is bounded on any closed subspace of the range $\mathcal{R}(A)$. If $\mathcal{N}(A)$ is uncomplemented in X, then $A_U^\dagger$ is always an unbounded operator.*

We emphasize that we can justify the central role of Definition 3 with respect to well-posedness concepts in this paper with the following corollary, which is an immediate consequence of Proposition 7.

Corollary 1. *For a bounded linear operator $A : X \to Y$ mapping between the infinite-dimensional Banach spaces X and Y, the operator equation* (1) *is well-posed in the sense of Definition 3 if and only if there is a bounded linear generalized pseudoinverse $A_U^\dagger : \mathcal{R}(A) \to U$ for a decomposition $X = \mathcal{N}(A) \oplus U$ of the Banach space X with null-space $\mathcal{N}(A)$.*

As in Lemma 1, the well-posed and strictly singular operators are those with finite-dimensional range.

Lemma 2. *Let A be well posed and strictly singular. Then $\mathcal{R}(A)$ is finite dimensional and A is also compact. Conversely, if $\mathcal{R}(A)$ is finite dimensional, then A is well-posed and strictly singular.*

Proof. If A is well-posed, the generalized pseudoinverse $A_U^\dagger$ is bounded. Since the strictly singular operator form an ideal, we have that $A_U^\dagger A$ is strictly singular, but this is the identity on U. Thus, U is finite dimensional and hence $\mathcal{R}(A) = A(U)$ is finite dimensional as well. Conversely, if the range is finite dimensional, the operator is compact and thus strictly singular and the range is closed. By the open mapping theorem, we have an isomorphism of $\mathcal{R}(A)$ to $X/\mathcal{N}(A)$. Thus, $X/\mathcal{N}(A)$ is finite dimensional, which means that $\mathcal{N}(A)$ has finite codimension. It is well-known that then $\mathcal{N}(A)$ is complemented (cf., e.g., [1, Proposition 11.6]), hence A is well-posed. □

Proposition 8. *Let $A : X \to Y$ be such that the operator equation* (1) *is ill-posed problem in the sense of Definition 3. Then the equation is ill-posed of type II if and only if A is strictly singular.*

Proof. If A is not ill-posed of type II, $\mathcal{R}(A)$ contains a closed infinite-dimensional subspace $M = A[X_1]$, where $\mathcal{N}(A)$ is complemented in the subspace X_1 of X. Then the complement X_2 is closed and $M = A[X_2]$. Now $A|_{X_2}$ is an injective operator onto M, and by the open mapping theorem there exists a bounded inverse on M. Thus, A is not strictly singular. If, conversely, A is not strictly singular, then there exists an infinite-dimensional closed subspace X_1 of X such that the restriction $A|_{X_1}$ of the operator A with $M = A[X_1]$ has a bounded inverse. Now $X_1 \cap \mathcal{N}(A|_{X_1}) = \{0\}$, hence $\mathcal{N}(A|_{X_1})$ is trivially complemented in $A^{-1}[M]$ and M is a closed subspace of Y. Since A was assumed to be ill-posed, this means that A is not ill-posed of type II. □

Figure 3 reflects the refined situation of Definition 3. At first glance, it seems that the differences between Figure 2 and Figure 3 are not really great. But that is only an appearance, because the strictly singular operators with infinite-dimensional range on the right part of Figure 3 contain cases of uncomplemented null-spaces (see Proposition 9 and Example 1 below).

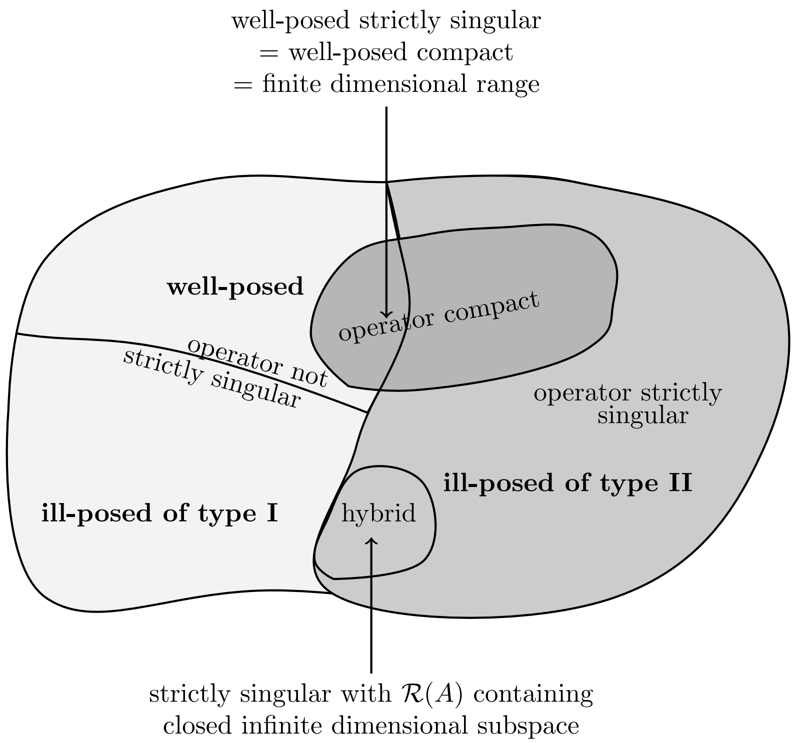

Figure 3: Case distinction for bounded linear operators between infinite-dimensional *Banach spaces* with *complemented* and *uncomplemented* null-spaces.

Beforehand we mention that uncomplemented null-spaces according to Definition 3 are not compatible with well-posed situations and ill-posed situations of type I, which are both assigned to the left part of Figure 3. Since the well-posed case (compact or not) is well-understood, we focus with respect to explanations of Figure 3 on the ill-posed operators, and especially on ill-posedness of type II.

We highlight, in particular, the hybrid case incorporated in Figure 3 as small circle (see also Example 1).

Definition 4. *In the setting of Definition 3, let us call an operator A of* hybrid *type when A is strictly singular and its range $\mathcal{R}(A)$ contains an infinite-dimensional closed subspace of Y.*

We label such cases as hybrid since their type is classified differently when switching between Definition 1 and Definition 3: They are type II by the latter definition but type I (or well-posed) by the former.

Next we verify that hybrid operators, which are ill-posed of type II, do not intersect the compact ones as indicated by the figure. Moreover, they are also not compatible with a complemented null-space.

Proposition 9. *An operator $A : X \to Y$ of hybrid type in the sense of Definition 4 cannot be compact, and its null-space is always uncomplemented.*

Proof. By definition $\mathcal{R}(A)$ contains a closed infinite-dimensional subspace M. Take a bounded sequence $(y_n) \subset M$ which does not have a convergent subsequence. Such a sequence exists since by the Riesz lemma since M is infinite dimensional. Even though A is non-injective, the open mapping theorem still gives a sequence in $x_n \in A^{-1}(y_n)$ with $\|x_n\| \leq C\|y_n\|$ and $Ax_n = y_n$. If A was compact, then y_n would have a convergent subsequence, which is a contradiction.

If the null-space would be complemented and A is not compact, then $X = \mathcal{N}(A) \oplus U$ is connected with an infinite-dimensional closed subspace U of X. Then this leads to a contradiction. Namely, let $M \subset \mathcal{R}(A)$ be an infinite-dimensional closed subspace of Y and, due to the continuity and injectivity of A on U, the set $A_U^\dagger[M]$ is an infinite-dimensional closed subspace of X on which A is bounded invertible as a consequence of Proposition 7. This, however, contradicts the strict singularity of A by Proposition 1. □

Example 1 (Hybrid, strictly singular, uncomplemented null-space)**.** Examples for strictly singular hybrid operators are the Mazur maps: Note that for any separable Banach space Y there exists a bounded map (Mazur map) $A : X = \ell^1 \to Y$ which is onto (cf. Example in [7]). Hence the range contains a closed infinite-dimensional space (namely Y itself). But by taking $Y = \ell^q$ with $1 < q < \infty$, this is a strictly singular operator by Proposition 3, hence hybrid. Due to Proposition 9 this operator is non-compact and its null-space is uncomplemented. We have here ill-posedness of type II by Proposition 8.

Example 2 (Non-compact, strictly singular, non-hybrid)**.** An example of a non-compact but strictly singular operator are the embeddings $\mathcal{E} : \ell^p \to \ell^q$ with $1 \leq p < q < \infty$. These are injective and strictly singular by Proposition 3. Since $\mathcal{N}(A) = \{0\}$ they cannot be hybrid as such instances do not exist in the injective case (cf. Figure 2). Moreover this operator is non-compact since we may consider the unit-sequence $(e_i)_{i=1}^\infty$ with has zero elements except at position i where the value is one. Clearly we have $\|e_i\|_{\ell^p} = 1$ and $\|e_i - e_j\|_{\ell^q} = 2^{1/q}$ for $i \neq j$ such that the bounded sequence $(e_i)_{i=1}^\infty$ cannot have a convergent subsequence. Hence the operator is non-compact, but ill-posed of type II.

Example 3 (General type I ill-posedness)**.** From any ill-posed operator of type II, we can construct a type I operator as follows by adding the identity:

Let $A : X \to Y$ be a ill-posed operator. Define

$$B : \ell^2 \times X \to \ell^2 \times Y$$
$$((a_n)_n, x) \to ((a_n)_n, Ax).$$

We have $\mathcal{N}(B) = (0, \mathcal{N}(A))$, $\mathcal{R}(B) = (\ell^2, \mathcal{R}(A))$. Thus, it is not difficult to see that $\mathcal{R}(B)$ is closed if and only if $\mathcal{R}(A)$ is. Moreover $\mathcal{N}(B)$ is complemented if and only if $\mathcal{N}(A)$ is. Thus, B is ill-posed by any of the above definitions, if and only if A is ill-posed of this type. However since $(\ell^2, 0)$ constitutes always a closed infinite-dimensional subspace in the range, where a bounded inverse exists, the operator B is not strictly singular and thus ill-posed of type I.

5 Alternative concepts of well-posedness

In the following, we consider bounded linear operators $A : X \to Y$ between Banach spaces X and Y and discuss further concepts of well-posedness from the literature, in comparison with the already given ones.

Definition 5. *We say that the operator equation $Ax = y$ (cf. (1)) is*

- *well-posed in the sense of Hadamard (HaWP) if and only if*

$$\mathcal{R}(A) = Y \text{ and } \mathcal{N}(A) = \{0\};$$

- *well-posed in the sense of Hofmann-Scherzer (HSWP) if and only if*

$$\mathcal{R}(A) \text{ is closed and } \mathcal{N}(A) = \{0\};$$

- *well-posed in the sense of Nashed-Votruba (NVWP) if and only if*

$$\mathcal{R}(A) \text{ closed },$$
$$\mathcal{N}(A) \text{ is complemented in } X, \text{ and}$$
$$\mathcal{R}(A) \text{ is complemented in } Y;$$

- *well-posed (WP) (cf. Def. 3) if and only if*

$$\mathcal{R}(A) \text{ closed and } \mathcal{N}(A) \text{ is complemented in } X;$$

- *Range-well-posed ($\mathcal{R}$-WP) (cf. Def. 1) if and only if*

$$\mathcal{R}(A) \text{ closed.}$$

The definition of (HSWP) was proposed in [10] (see also [4]) in a slightly different way as the requirement $N(A) = \{0\}$ together with the continuity property $Ax_n \to Ax \Rightarrow x_n \to x$. A consequence of the open mapping theorem is that (HSWP) problems have a continuous linear inverse on its range, such that the latter continuity property is equivalent to that listed in Definition 5. This approach, which reflects precisely the situation in Figure 2, was motivated by the concept of *local well-posedness* for nonlinear operator equations introduced in [10, Def. 1.1]. As also discussed in [9], for linear operator equations, the local character disappears, because local well-posedness takes place everwhere or nowhere.

The definition of (NVWP) is motivated by the epic study of generalized inverses in [16, 18], where the stated condition is equivalent to the existence of a bounded inner inverse mapping $Y \to X$, where an inner inverse is a linear operator B satisfying $ABA = A$. Compared to the well-posedness concept in Definition 3, the complementedness of the range is required as additional condition such that (NVWP) serves as a stronger condition for well-posedness.

Example 4. Let M be a closed subspace that is not complemented in $X = M \oplus N$. Consider the embedding operator $M \to X$.

$$\begin{aligned} A : M &\to X \\ x &\to x. \end{aligned}$$

We have that $\mathcal{N}(A) = \{0\}$, $\mathcal{R}(A) = M$ is closed. Thus, this is a well-posed problem by our definition. However, since the range is uncomplemented, it is (NV)-ill-posed.

Note that the open mapping theorem shows that Hadamard well-posedness (HaWP) of Definition 5 also implies stability such that we are in accordance with the classical Hadamard definition. Together with our definitions above, this constitutes five instances for definitions of well-posedness.

Clearly the following implications hold true:

$$\text{(HaWP)} \quad \Rightarrow \quad \begin{matrix} \text{(NVWP)} \\ \text{(HSWP)} \end{matrix} \quad \Rightarrow \quad \text{(WP)} \quad \Rightarrow \quad (\mathcal{R}\text{-WP}).$$

If X, Y are Hilbert space, then all closed subspaces are complemented and most of the definitions are equivalent:

$$\text{(NVWP)} \quad \Longleftrightarrow \quad \text{(WP)} \quad \Longleftrightarrow \quad (\mathcal{R}\text{-WP})$$

The same equivalences hold in the Banach space case if A is injective with dense range.

Remark 1. For $y \in \mathcal{R}(A)$ and $y_n \in Y$ with $\|y_n - y\|_Y \to 0$ as $n \to \infty$ the *quasi-distance*

$$\operatorname{qdist}(A^{-1}[y_n], A^{-1}[y]) := \sup_{\tilde{x} \in A^{-1}[y_n]} \inf_{x \in A^{-1}[y]} \|\tilde{x} - x\|_X,$$

first introduced in [11] for general nonlinear problems, is also one possible tool to evaluate stability for linear problems with non-trivial null-spaces $\mathcal{N}(A)$ of the linear operator A mapping between Banach spaces X and Y. The monograph [4] outlines in Section 1.2 the interplay between range-well-posedness ($\mathcal{R}$-WP) and this kind of stability (labeled in [4] as well-posedness in the sense of Ivanov) in the case of multiple solutions (and in a nonlinear setup). Propositions 1.10 and 1.12 in [4] show that ($\mathcal{R}$-WP) implies $\text{qdist}(A^{-1}[y_n], A^{-1}[y]) \to 0$ as $n \to \infty$. If ($\mathcal{R}$-WP) fails and the null-space is complemented, then this kind of stability also fails. When A is a linear operator between Banach space as in this article, we may observe that qdist gives the norm $\|A^{-1}[y_n] - A^{-1}[y]\|_{X/\mathcal{N}(A)}$ in the quotient space $X/\mathcal{N}(A)$; thus this stability requirement is equivalent to stability in the quotient space which is equivalent to ($\mathcal{R}$-WP); see the next Theorem 1. This has already been observed in [9] for the Hilbert space case.

5.1 Characterizations of well-posedness by generalized inverses

Next, we characterize the well-posedness in Definition 5 and the previously given ones in terms of generalized inverses, building on the seminal work of Nashed and Votruba [18]. We denote by $L(X,Y)$ the bounded maps from $X \to Y$ and consider only bounded operators $A : X \to Y$.

Theorem 1. *A problem $Ax = y$ is*

1. *(HaWP) if and only if*

$$\exists B \in L(Y,X) \text{ with } BA = Id_X, \quad AB = Id_Y.$$

2. *(HSWP) if and only if*

$$\exists B \in L(\mathcal{R}(A), X) \text{ with } BA = Id_X.$$

3. *(NVWP) if and only if*

$$\exists B \in L(Y,X) \text{ with } ABA = A.$$

4. *(WP) if and only if*

$$\exists B \in L(\mathcal{R}(A), X) \text{ with } ABA = A.$$

5. *($\mathcal{R}$-WP) if and only if*

$$\exists \text{(possibly nonlinear) continuous map } B : \overline{\mathcal{R}(A)} \to X \text{ with } AB \circ A = A.$$

or, equivalently,

$$\tilde{A} : X/\mathcal{N}(A) \to \mathcal{R}(A) \text{ has a bounded linear inverse.}$$

In the last item we understand by $\tilde{A}$ the operator defined on the quotient space $\tilde{A}\langle x\rangle = Ax$, with $\langle x\rangle$ the equivalence class in $X/\mathcal{N}(A)$, which is clearly a well-defined operator.

Proof. 1. is classical with $B = A^{-1}$. Boundedness of B is given by the open mapping theorem.

Consider 2.: If (HSWP) holds, then $A : X \to \mathcal{R}(A)$ is injective and surjective, and by the open mapping theorem, a continuous right inverse $B : \mathcal{R}(A) \to X$ with $BAx = x$ exists. Conversely, if such a map exists, then $\mathcal{N}(A) = \{0\}$ and convergence of a sequence $Ax_n \to y$ induces convergence of $x_n = BAx_n \to z$. Hence $y = Az$, and the range is closed.

The equivalence of 3. is proven by Nashed specializing the work of Nashed and Votruba [18]; see [16, Theorem 7.1].

4. If (WP) holds, then by considering A mapping to the Banach space $\mathcal{R}(A)$, we may again use [16, Theorem 7.1] to show that B exists and is bounded. Conversely, if such a B exists, then by the same theorem $\mathcal{N}(A)$ must be complemented. We show that the range of A is closed: We may extend B by continuity to $\overline{\mathcal{R}(A)}$. Let a sequence satisfy $Ax_n \to y$. Then, by continuity $BAx_n \to By =: w$ and by definition of B as inner inverse $ABAx_n \to y$. Hence by continuity of A we have $y = Aw$, which implies that the range of A is closed.

Concerning the second conclusion in 5. This is simply the open mapping theorem applied to the injective operator $\tilde{A}$. That the range of A is closed if $\tilde{A}$ has a bounded inverse follows easily since convergence of $Ax_n \to y$, implies by $Ax_n = \tilde{A}\langle x_n \rangle$ convergence of $\tilde{A}\langle x_n \rangle$ and thus convergence of $\langle x_n \rangle$. Consequently, y is in $\mathcal{R}(A) = \mathcal{R}(\tilde{A})$.

Regarding the first equivalence in 5. Assume that such a B exists. Then as in the proof of 4. we can conclude that $\mathcal{R}(A)$ is closed since linearity is not needed there. The opposite direction requires a corollary of the Bartle-Graves theorem, Theorem 2, stated below: Assume that $\mathcal{R}(A)$ is closed. Then $\mathcal{R}(A)$ is a Banach space and $A : X \to \mathcal{R}(A)$ is surjective. Hence a continuous right-inverse exists: $AB(y) = y$ for all $y \in \mathcal{R}(A)$. But this is exactly the definition of an inner inverse: $AB \circ A = A$. □

The powerful Bartle–Graves corollary that we used in the last part reads as follows (cf. [3, Corollary 5G.4, p. 298]):

Theorem 2. *Let $A : X \to Y$ be a bounded surjective map between Banach spaces X and Y. Then there is a (in general nonlinear) continuous right-inverse B, i.e., we have*

$$AB(y) = y \qquad (\forall y \in Y).$$

From an operator point of view, the difference between the well-posedness and $\mathcal{R}$-well-posedness definitions is essentially that of whether the problem allows for a linear or only a nonlinear continuous solution mapping on its range. The difference to the (NV) concepts is that of attainability; in (WP) we only consider $y \in \mathcal{R}(A)$ as possible data, while (NVWP) allows for any $y \in Y$ as data.

As an illustration we may characterize the different well-posedness concepts in the following table:

Typ of WP	Range characterization	Operator characterization
(HaWP)	$\mathcal{R}(A) = Y$ $\mathcal{N}(A) = \{0\}$	$\exists B \in L(Y, X)$ $BA = Id_X$ $AB = Id_Y$
(HSWP)	$\mathcal{R}(A)$ closed $\mathcal{N}(A) = \{0\}$	$\exists B \in L(\mathcal{R}(A), X)$ $BA = Id_X$
(NVWP)	$\mathcal{R}(A)$ closed $\mathcal{N}(A)$ complemented $\mathcal{R}(A)$ complemented	$\exists B \in L(Y, X)$ $ABA = A$
(WP) (Def 3)	$\mathcal{R}(A)$ closed $\mathcal{N}(A)$ complemented	$\exists B \in L(\mathcal{R}(A), X)$ $ABA = A$
$\mathcal{R}$-WP (Def 1)	$\mathcal{R}(A)$ closed	$\exists B : C(\mathcal{R}(A), Y)$ B possibly nonlinear $ABA = A$

5.2 Alternatives concepts of types of ill-posedness

The intuition behind "type I ill-posedness" is that an operator contains an infinite-dimensional *well-posed subproblem.* As we have seen, there are various concepts of well-posedness, hence this analogously leads to different concepts of "type I ill-posedness", depending on what kind of well-posedness we consider for the subproblem. Thus, we may introduce a third alternative type distinction (besides those in Definitions 3 and 1) based on the (NVWP) concept of well-posedness:

Definition 6. *An ill-posed operator $A : X \to Y$ is (NV)-type I ill-posed whenever there exists a closed infinite-dimensional subspace $M \subset \mathcal{R}(A)$, such that*

$$\mathcal{N}(A) \text{ is complemented in } A^{-1}[M]$$

and

$$M \text{ is complemented in } Y,$$

and we call it (NV)-type II otherwise.

Clearly, (NV)-type I ill-posedness implies our type I ill-posedness definition from Definition 3, since the complemented range of M is required additionally. Thus, we have the following implication within the ill-posed operators:

$$\text{(NV)-type I} \subset \text{type I} \subset \mathcal{R}\text{-type I}.$$

In particular, the class (NV)-type I ill-posedness does not contain all ill-posed operators that are not strictly singular, but leaves some out. Such operators that are (NV)-type II ill-posed but of type I (not strictly singular) (i.e. the set "type I \ (NV)-type I") we label as (NV)-hybrid class of operators:

Definition 7. *The class of ill-posed operators that are (NV)-type II but type I (non-strictly singular) we call (NV)-hybrid operators.*

Equivalently, these operators are characterized by the fact that there is an infinite-dimensional closed subspace X_1 such that $A|_{X_1}$ has a bounded inverse on its range but where $\mathcal{R}(A|_{X_1})$ is not complemented in Y. Moreover for all infinite-dimensional subspaces spaces X_1 which allow for such a bounded inverse of $A|_{X_1}$, the subspace $\mathcal{R}(A|_{X_1})$ is not complemented in Y.

The full picture is now as in Figure 4: Depending on which choice of type-

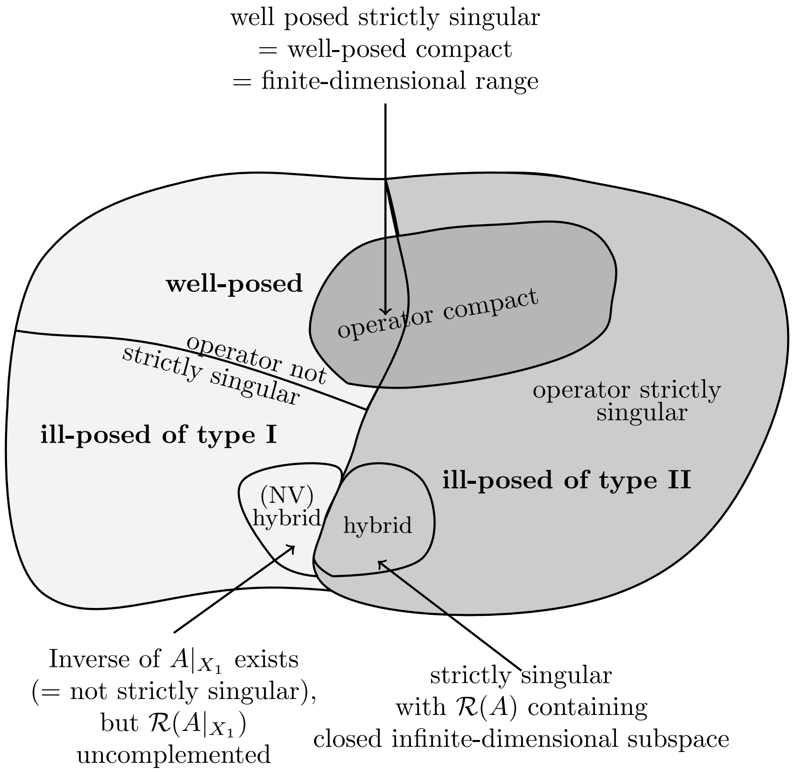

Figure 4: Position of hybrid operators.

distinction one chooses, the hybrid cases can be classfied differently as type I or type II (or well-posed).

Remark 2. Let us emphasize that our terminology of well-/ill-posedness and type I/type II is idiosyncratic as there does not seem to exist universally accepted definitions in the literature. In fact, various authors simply call "ill-posed" any of the above ill-posedness concepts without a further identifier.

Since the problem distinction by types is based on a well-posedness concept of a subspace, the same holds true for this: depending on what one calls well-posed, concrete problems might be attributed to different classes. Our definition of type I (cf. Definition 3) above is that of Nashed in [17, Theorem 4.5], and it is also used by Flemming in [4]. However, let us mention that Nashed (e.g. in [17, Theorem 4.4]) (preferably) uses the "(NV)-type I" version. On the other hand, in the same article, Nashed apparently uses a well-posedness concept (although never explicitly mentioned) based on $\mathcal{R}$-well-posedness, which in this combination seems to be slightly inconsequent. A similar setup ($\mathcal{R}$-well-posedness but type I from Definition 3) is Flemming's well-posedness concepts in [4].

In the non-injective situations we have three definitions of well-posedness (WP, NVWP, $\mathcal{R}$-WP), and each of this definitions could be paired with a type-distinction using one of these definitions (a la "contains a (WP, NVWP, $\mathcal{R}$-WP) infinite-dimensional subproblem"), which would give us a 3x3 matrix of possible classification schemes in the Banach space case, where in each situation the borders between types and well-posedness in a picture like Figure 4 would be different (as for our hybrid cases).

We decided for the concept of (WP) from Definition 3 because of the characterization of both, on the one hand the well-posedness by the existence of a bounded generalized pseudoinverse, and on the other hand the equivalence of ill-posed strict singular cases with type II problems. Moreover note that our type distinction is consistent with the well-posedness definition and does not mix different well-posedness concepts.

Remark 3. The definition of (NV)-hybrid operators is motivated by Nashed's article, where mainly the (NV)-type I class was considered. Although the definition in [17, Theorem 4.5] seems to suggest that type I and (NV)-type I are equivalent, this is not the case since in this theorem the complemented condition on M was forgotten (but implicitly used) and should be amended there. In his paper [17], Nashed also states that "type I is not identical with non-strictly singular", which clearly indicates that he is using the (NV)-type I definition, and the class of operators which are non-strictly singular but Nashed's type I are precisely our (NV)-hybrid class.

Nashed also gave an example of operators that are never (NV)-type I (the proof follows [17, p. 69f]). Compare the next proposition with Prop. 4, according to which any operator $c_0 \to \ell^p$, $p \in [1, \infty)$, is never type I:

Proposition 10 (Nashed, Schock)**.** *A bounded operator $A : c_0 \to \ell^\infty$ is never (NV)-type I.*

Proof. Let M be an infinite-dimensional subspace of $\mathcal{R}(A)$ which is complemented in ℓ^∞ and where X_1 defined as $M = AX_1$ has a complement in $A^{-1}[M]$. If such an M exists, then A would be (NV)-type I, and we show that M cannot exists. Let $\tilde{X}_1$ be the complement of $\mathcal{N}(A)$ in X_1 such that $M = \mathcal{R}(A|_{\tilde{X}_1})$ and $A : \tilde{X}_1 \to M$ is injective (and surjective). Thus, $A|_{\tilde{X}_1}$ has an bounded inverse $S := (A|_{\tilde{X}_1})^{-1} : M \to \tilde{X}_1$, and $\tilde{X}_1$ must be infinite dimensional (since M is so). Now since M is complemented, there exists a continuous projection

$Q : \ell^\infty \to M$. The mapping $SQA : c_0 \to \tilde{X}_1 \subset c_0$ is the identity on $\tilde{X}_1$ and hence a continuous projection $c_0 \to \tilde{X}_1$. This means that $\tilde{X}_1$ is complemented in c_0. It is known [14, Theorem 2.a.3] that then $\tilde{X}_1$ is isomorphic to c_0; denote the isomorphism by $i : \tilde{X}_1 \to c_0$. It follows that $iSQAc_0 \to c_0$ is surjective since SQA maps surjective to $\tilde{X}_1$. It is known [2, p. 114] that any bounded map $\ell^\infty \to c_0$ is weakly compact and weakly compact operators form an ideal, hence SQA is weakly compact. Thus, by the isomorphism i, any bounded set in c_0 is weakly compact. By the Eberlein-Smulian theorem [2, p. 18], this is equivalent to c_0 being reflexive, which is a contradiction. □

Thus, such mappings from $c_0 \to \ell^\infty$ are either well-posed or (NV)-type II. For instance the embedding $\mathcal{E} : c_0 \to \ell^\infty$ is injective with closed range c_0 and thus well-posed by our definition; it follows from the proof of Proposition 10 that it is (NV)-ill-posed as c_0 cannot be complemented in ℓ^∞.

A similar reasoning allows us to construct an (NV)-hybrid operator:

Example 5 ((NV)-hybrid)**.** Consider the following operator:

$$\begin{aligned} B : c_0 \times \ell^2 &\to \ell^\infty \times \ell^2 \\ (x, y) &\to (x, Ay), \end{aligned}$$

where $A : \ell^2 \to \ell^2$ is a compact injective operator. Thus, B is the Cartesian product of the embedding operator $\mathcal{E} : c_0 \to \ell^\infty$ and a compact operator A mapping in the Hilbert space ℓ^2. We have that $\mathcal{N}(B) = \{0\}$ and $\mathcal{R}(B) = (c_0, \mathcal{R}(A))$. Since c_0 is closed, $\mathcal{R}(B)$ is closed if and only if $\mathcal{R}(A)$ is closed, which is not the case, hence B constitutes an ill-posed operator by our Definition 3. Since the subspace $(c_0, 0)$ allows for a continuous inverse on this set, the operator is not strictly singular, thus of type I. We shows that it is not (NV)-type I: Consider an infinite-dimensional closed subspace $M = B(X_1)$, $X_1 \subset c_0 \times \ell^2$ such that M is complemented. We have $M = (M_1, M_2)$ is closed. Let $N = (N_1, N_2)$ be the topological complement of M in $\ell^\infty \times \ell^2$. Then N is closed and this implies that N_1 and N_2 are closed. Thus, N_1 constitutes a topological complement of M_1 in ℓ^∞. We also have to show that M_1 is infinite dimensional. Since M is so, M_1 is finite dimensional only if M_2 is infinite dimensional. However, M_2 is a closed subspace in the range of the compact operator A and hence can be at most finite dimensional. Thus, M_1 must be infinite dimensional.

As above this now allows for a continuous projection $Q : \ell^\infty \to M_1$, and combining this with the inverse of the first component of $B|_{(M_1,0)} \to (M_1, 0)$, we arrive at a weakly compact map to $M_1 \subset c_0$ where M_1 is complemented in c_0 by the same reasoning as in Proposition 10. This again yields the contradiction that c_0 is reflexive. Thus, no such subset X_1 can exist, which means that B cannot be of (NV)-type I.

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