Ergänzungen IV zur Vorlesung Inverse Probleme Sommersem. 2018

Über den Stetigkeitsmodul von A^{-1}

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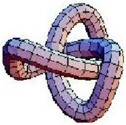

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Einige Literaturhinweise:

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▷ FLEMMING, J., HOFMANN, B., MATHÉ, P.: Sharp converse results for the regularization error using distance functions. *Inverse Problems* **27** (2011), 925006.

▷ B. HOFMANN; S. KINDERMANN: On the degree of ill-posedness for linear problems with non-compact operators. *Methods and Applications of Analysis* **17** (2010), No. 4.

▷ B. HOFMANN, P. MATHÉ, M. SCHIECK: Modulus of continuity for conditionally stable ill-posed problems in Hilbert space. *J. Inv. Ill-posed Problems* **16** (2008), pp. 567-585.

Ill-posed linear operator equations

X, Y infinite dimensional separable Hilbert spaces
with norms $\|\cdot\|$ and inner products $\langle\cdot,\cdot\rangle$.

$A: X \longrightarrow Y$ **injective bounded** linear forward operator
with **non-closed range**, i.e., $\mathcal{R}(A) \neq \overline{\mathcal{R}(A)}$.

We consider the **ill-posed** linear operator equation

$$\boxed{A\,x \;=\; y \qquad (x \in X,\; y \in Y)\,. \qquad\qquad (*)}$$

$A^{-1}: \mathcal{R}(A) \subset Y \to X$ is **unbounded (not continuous)** and

$\sup\{\|x\|:\; x \in X,\; \|Ax\| \le \delta\} = \infty$ even if $\delta > 0$ is small.

The solution theory of ill-posed problems is preferably based on the fact that these problems become

conditionally well-posed

after imposing certain **smoothness assumptions** by restricting the admissible solutions to a set $\mathfrak{M}$.

Then the **severity of the ill-posedness** in solving a problem $(*)$ depends on the interplay between the smoothing properties of the operator A and the smoothness of potential solutions $x \in \mathfrak{M} \subset X$ measured by the **modulus of continuity**

$$\omega(A, \mathfrak{M}, \delta) := \sup\{\|x\| : \; x \in \mathfrak{M}, \; \|Ax\| \le \delta\} < \infty, \quad \delta > 0.$$

We associate A with the positive self-adjoint operator

$$H := A^*A : X \to X$$

and set $a := \|H\| = \|A\|^2$.

Then we have for the spectrum $\sigma(H)$ of H

$$0 = \min\{s \in \mathbb{R} : s \in \sigma(H)\}, \qquad a = \max\{s \in \mathbb{R} : s \in \sigma(H)\}.$$

NASHED distinguishes for $(*)$

ill-posedness of **type I** where A is **non-compact**

and

ill-posedness of **type II** where A is **compact.**

For the analysis of ill-posed problems **solution smoothness** is often measured relative to the operator A governing the equation $(*)$.

One can quantify the **individual smoothness** of an element $x \in X$ with respect to H by using the **point-wise** spectral information, i.e., the **distribution function**

$$F_x^2(t) := \|E_t x\|^2 := \langle \chi_{(0,t]}(H)\, x, x \rangle = \|\chi_{(0,t]}(H) x\|^2, \quad 0 < t < \infty,$$

where $\chi_{(0,t]}$ is the characteristic function on the interval $(0, t]$. This idea goes back to A. NEUBAUER. See also

▷ FLEMMING, J., HOFMANN, B., MATHÉ, P.: Sharp converse results for the regularization error using distance functions. *Inverse Problems* **27** (2011), 925006.

Proposition (Properties of the distribution function)

- F_x^2 is non-decreasing and right-continuous;
- $\lim\limits_{t\to 0} F_x^2(t) = 0$ due to the ill-posedness of $(*)$;
- $F_x^2(t) = \|x\|^2$ for $a \le t < \infty$;
- $F_x^2(t) = \int\limits_0^t d\,\|E_s x\|^2$ for $t > 0$, where $E_t = E_t(H)$ denotes the spectral resolution of the operator H;
- We have $\|h(H)x\|^2 = \int\limits_0^a h^2(t)\, d\,\|E_t x\|^2 = \int\limits_0^a h^2(t)\, dF_x^2(t)$ for any bounded measurable real function h;
- F_x^2 is a step function for compact H (discrete spectrum), but a continuous function if H possesses a purely continuous spectrum.

The traditional way of quantifying solution smoothness uses **smoothness classes** in terms of **source sets** $\mathfrak{M} = \mathcal{M}_{\varphi,R}$:

$$\mathcal{M}_{\varphi,R} := \{x \in X : \; x = \varphi(H)\, v, \; v \in X, \; \|v\| \le R\}, \qquad R > 0.$$

Above, the functions $\varphi : (0, a] \to (0, \infty)$ are derived from variable Hilbert scales and called **index functions** (non-decreasing with $\lim\limits_{t \to +0} \varphi(t) = 0$). Here we follow the concept of MATHÉ/PEREVERZEV.

Source sets express the solution smoothness with respect to the spectrum of H in an **integral** manner, since we have that

$$x \in \mathcal{M}_{\varphi,R} \quad \text{if and only if} \quad \int_0^a \frac{1}{\varphi^2(t)}\, dF_x^2(t) \le R^2.$$

Alternatively, one can assign **smoothness classes** by considering the **level sets** $\mathfrak{M} = \mathcal{E}_{\psi,E}$ defined as

$$\mathcal{E}_{\psi,E} := \{x \in X : \ F_x^2(t) \le E^2\, \psi^2(t),\ 0 < t \le a\}, \quad E > 0,$$

for index functions $\psi : (0, a] \to (0, \infty)$.

Proposition (Properties of source sets and level sets)

1. For arbitrary index functions φ and ψ the sets $\mathcal{M}_{\varphi,R}$ and $\mathcal{E}_{\psi,E}$ are **centrally symmetric** and **convex.**
2. If the operator A is compact then the sets $\mathcal{M}_{\varphi,R}$ and $\mathcal{E}_{\psi,E}$ are **compact.**
3. For $0 < \kappa < 1$ we have $\mathcal{E}_{\psi,E} \subset \mathcal{M}_{\psi^\kappa,E_\kappa}$ with $E_\kappa^2 = \frac{E^2(\psi(a))^{2(1-\kappa)}}{1-\kappa}$, and we have $\mathcal{M}_{\psi,E} \subset \mathcal{E}_{\psi,E}$ if φ and ψ coincide.

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We are going to study the modulus of continuity $\omega(A, \mathfrak{M}, \delta)$ with focus on the sets $\mathcal{M}_{\varphi,R}$ and $\mathcal{E}_{\psi,E}$ and set

$$U\,\mathfrak{M} := \{z \in Z: \; z = U\,x, \;\; x \in \mathfrak{M}\},$$

for linear operators $U\colon X \to Z$ and some Hilbert space Z.

In that sense, we use

$$K\,\mathfrak{M} := \{x \in X: \; x = K\,\tilde{x}, \;\; \tilde{x} \in \mathfrak{M}\}$$

for constants $K > 0$ by identifying the constant K with the multiple KI of the unit operator.

Proposition (Calculus for the modulus of continuity)

For centrally symmetric and convex sets $\mathfrak{M}$, which means that with $x_1, x_2 \in \mathfrak{M}$ also the elements $-x_2$ and $(x_1 - x_2)/2$ belong to $\mathfrak{M}$, the following properties hold for the moduli of continuity:

(a) If $\mathfrak{M}$ is bounded then $\omega(A, \mathfrak{M}, \delta)$ is a finite, positive and non-decreasing function for $\delta > 0$ and it is constant for $\delta \geq \bar{\delta} := \sup_{x \in \mathfrak{M}} \|Ax\|$.

(b) If $\mathfrak{M}$ is relatively compact then $\lim_{\delta \to 0} \omega(A, \mathfrak{M}, \delta) = 0$.

(c) $\omega(A, K\mathfrak{M}, \delta) = K\, \omega(A, \mathfrak{M}, \delta/K)$ for $K > 0$.

(d) $\omega(A, \mathfrak{M}, C\delta) \leq C\, \omega(A, \mathfrak{M}, \delta)$ for $C > 1$.

(e) $\omega(A, K\mathfrak{M}, C\delta) \leq \max\{C, K\}\, \omega(A, \mathfrak{M}, \delta)$ for $C, K > 0$.

(f) The decay rate of $\omega(A, \mathfrak{M}, \delta) \to 0$ as $\delta \to 0$ is at most linear.

For source sets and level sets representing $\mathfrak{M} \subset X$ the **best possible error for reconstruction** of the solution $x \in \mathfrak{M}$ based on noisy data y^δ, satisfying $\|y^\delta - y\| \leq \delta$ instead of the exact right-hand side $y \in \mathcal{R}(A)$, is given by

$$\tilde{\omega}(A, \mathfrak{M}, \delta) := \sup \{\|x_1 - x_2\| : \ x_1, x_2 \in \mathfrak{M}, \ \|A(x_1 - x_2)\| \leq \delta\}.$$

For centrally symmetric and convex sets $\mathfrak{M}$ we have

$$\omega(A, \mathfrak{M}, \delta) \leq \tilde{\omega}(A, \mathfrak{M}, \delta) \leq \omega(A, 2\mathfrak{M}, \delta), \qquad \delta > 0.$$

Consequently, $\omega(A, \mathfrak{M}, \delta)$ acts as a measure of ill-posedness pre-estimating the reconstruction error in solving $(*)$ for given $\delta > 0$.

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Recent results concerning the concavity of the modulus of continuity

Because of $\mathcal{M}_{\varphi,R} = R\,\mathcal{M}_{\varphi,1}$ and $\mathcal{E}_{\psi,E} = E\mathcal{E}_{\psi,1}$ it is sufficient to focus on $R = E = 1$.

Hence for simplicity we set

$$\mathcal{M}_\varphi := \mathcal{M}_{\varphi,1} \qquad \text{and} \qquad \mathcal{E}_\psi := \mathcal{E}_{\psi,1}$$

and consider in the sequel only these sets.

Based on results from IVANOV/KOROLYUK 1969 it was formulated after the millennium by MATHÉ/PEREVERZEV 2003:

Proposition (Concavity for type II under additional cond.)

For **compact** A with decreasingly ordered singular values $\sigma_n \to 0$ as $n \to \infty$ and under a **concavity** requirement on

$$\varphi^2((\Theta^2)^{-1}(t)) \quad \text{with} \quad \Theta(t) := \sqrt{t}\,\varphi(t), \quad 0 < t \le \Theta^2(a),$$

the function

$$\tau(A, \mathcal{M}_\varphi, \delta) := \omega^2(A, \mathcal{M}_\varphi, \sqrt{\delta}), \quad \delta > 0,$$

is a **concave linear spline** $s(\delta)$, or more precisely the smallest concave index function, interpolating

$$s(\Theta^2(\sigma_n^2)) = \varphi^2(\sigma_n^2), \quad n = 1, 2, \ldots .$$

Later in 2008 by H./MATHÉ/SCHIECK the following extension to the general case of type II ill-posedness was proven:

Proposition (Concavity for type II in general)

For all **compact** operators A the function

$$\tau(A, \mathcal{M}_\varphi, \delta) := \omega^2(A, \mathcal{M}_\varphi, \sqrt{\delta}), \quad \delta > 0,$$

is **concave**.

The type I ill-posedness covering the case of multiplication operators with multiplier functions having an essential zero was left open.

Note that the modulus function $\omega(A, \mathcal{M}_\varphi, \delta)$ itself need not be concave for sufficiently large $\delta \geq \delta_0 > 0$.

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New results on concavity proof techniques

The goal of the joint research at WIAS Berlin in March 2011 was to extend the concavity result for $\tau(A, \mathcal{M}_\varphi, \delta)$ to type I ill-posedness and moreover to consider $\tau(A, \mathcal{E}_\psi, \delta)$.

Main Theorem

For **every** bounded linear operator $A : X \to Y$ with non-closed range $\mathcal{R}(A)$ and arbitrary index functions φ and ψ defined on the interval $(0, a]$ the functions

$$\tau(A, \mathfrak{M}, \delta) := \omega^2(A, \mathfrak{M}, \sqrt{\delta}), \quad \delta > 0,$$

are **concave** for both classes $\mathfrak{M} := \mathcal{M}_\varphi$ and $\mathfrak{M} := \mathcal{E}_\psi$.

Note that due to the identity $\omega\,(A, K\mathfrak{M}, \delta) = K\,\omega\,(A, \mathfrak{M}, \delta/K)$ for $K > 0$ the concavity of the main theorem carries over to the functions $\omega^2(A, \mathcal{M}_{\varphi,R}, \sqrt{\delta})$ and $\omega^2(A, \mathcal{E}_{\psi,E}, \sqrt{\delta})$, respectively, for all $\delta > 0$, $E > 0$ and $R > 0$.

For the proof of the main theorem we need the spectral theorem and some more auxiliary results.

Spectral Theorem (for bounded self-adjoint operators)

For every bounded self-adjoint linear operator $H : X \to X$ mapping in the the separable Hilbert space X there exist a measurable space $(\Omega, \mathcal{A}, \mu)$, a unitary transformation $U : X \to Z := L^2(\Omega, \mathcal{A}, \mu)$, and a measurable function $f : \Omega \to \sigma(H) \setminus \{0\} \subseteq (0, \|H\|] \subset \mathbb{R}$ such that $M_f := UHU^*$ is a multiplication operator defined as

$$[M_f\, h](\omega) := f(\omega)\, h(\omega), \quad \omega \in \Omega,$$

and mapping Z into itself. Moreover we have $\eta(H) = U^* M_{\eta(f)} U$ for bounded measurable functions η.

Proposition (Modulus under unitary transformations)

(i) We have that

$$\omega\left(A, \mathfrak{M}, \delta\right) = \omega\left(H^{1/2}, \mathfrak{M}, \delta\right), \quad \delta > 0.$$

(ii) If $B = UGU^* : Z \to Z$ for some unitary operator $U : X \to Z$ mapping into the Hilbert space Z with norm $\|\cdot\|_*$ and some bounded linear operator $G : X \to X$, then

$$\omega\left(G, \mathfrak{M}, \delta\right) = \omega\left(B, U\mathfrak{M}, \delta\right), \quad \delta > 0.$$

Corollary

For $A^*A = U^*M_fU$ we have that

$$\omega\left(A, \mathfrak{M}, \delta\right) = \omega\left(M_{f^{1/2}}, U\mathfrak{M}, \delta\right), \quad \delta > 0.$$

It is interesting to determine the analogs of $\mathcal{M}_{\varphi,R}$ and $\mathcal{E}_{\psi,E}$ in the multiplication context, i.e., the images $U\mathcal{M}_{\varphi,R}$ and $U\mathcal{E}_{\psi,E}$.

To do so we apply the corollary.

For $H = U^* M_f U$ we shall abbreviate $\|g\|_* := \|g\|_{L_2(\Omega,\mathcal{A},\mu)}$

Lemma

$$U\mathcal{M}_{\varphi,R} = \{g \in Z : \ \ g = \varphi(f)h, \ \|h\|_* \leq 1\}.$$

$$U\mathcal{E}_{\psi,E} = \left\{ g \in Z : \int_{0<f(\omega)\leq t} |g(\omega)|^2 \ d\mu(\omega) \leq E^2\psi^2(t) \right\}.$$

Proof ideas for the main theorem

We carry out the proof for $\mathfrak{M} := \mathcal{M}_\varphi$ in detail:
For the function $\Theta(t) := \sqrt{t}\,\varphi(t),\ 0 < t \le a$, we find that

$$\tau(A, \mathcal{M}_\varphi, \delta) = \sup\left\{ \|g\|_*^2 :\ g = \varphi(f)h,\ \|h\|_* \le 1,\ \|\sqrt{f} g\|_*^2 \le \delta \right\}$$
$$= \sup\left\{ \|\varphi(f)h\|_*^2 :\ \|h\|_* \le 1,\ \|\Theta(f)h\|_*^2 \le \delta \right\}.$$

Consider arbitrarily chosen $0 < \delta_1 < \delta < \delta_2$ and
$\delta = \lambda\delta_1 + (1-\lambda)\delta_2$ for some appropriate $0 < \lambda < 1$.

With given $\varepsilon > 0$ we can find elements $h_1, h_2 \in L^2(\Omega, \mathcal{A}, \mu)$, $\|h_1\|_* \le 1, \|h_2\|_* \le 1$, satisfying

$$\int_0^a \Theta^2(f(\omega))h_1^2(\omega)\, d\mu(\omega) \le \delta_1, \quad \int_0^a \varphi^2(f(\omega))h_1^2(\omega)\, d\mu(\omega) \ge \tau(A, \mathcal{M}_\varphi, \delta_1) - \varepsilon,$$

$$\int_0^a \Theta^2(f(\omega))h_2^2(\omega)\, d\mu(\omega) \le \delta_1, \quad \int_0^a \varphi^2(f(\omega))h_2^2(\omega)\, d\mu(\omega) \ge \tau(A, \mathcal{M}_\varphi, \delta_2) - \varepsilon.$$

We let h be chosen such that

$$h^2(\omega) := \lambda h_1^2(\omega) + (1-\lambda)h_2^2(\omega), \quad \omega \in \Omega. \tag{1}$$

Plainly, $\|h\|_* \le 1$. Also we have that

$$\int_0^a \Theta^2(f(\omega))h^2(\omega)\, d\mu(\omega)$$
$$= \lambda \int_0^a \Theta^2(f(\omega))h_1^2(\omega)\, d\mu(\omega) + (1-\lambda)\int_0^a \Theta^2(f(\omega))h_2^2(\omega)\, d\mu(\omega)$$
$$\le \lambda\delta_1 + (1-\lambda)\delta_2 = \delta.$$

Therefore we conclude that

$$\tau(A, \mathcal{M}_\varphi, \delta) \geq \int_0^a \varphi^2(f(\omega))h^2(\omega)\, d\mu(\omega)$$

$$= \lambda \int_0^a \varphi^2(f(\omega))h_1^2(\omega)\, d\mu(\omega) + (1-\lambda) \int_0^a \varphi^2(f(\omega))h_2^2(\omega)\, d\mu(\omega)$$

$$\geq \lambda\, \tau(A, \mathcal{M}_\varphi, \delta_1) + (1-\lambda)\, \tau(A, \mathcal{M}_\varphi, \delta_2) - \varepsilon.$$

Letting $\varepsilon \to 0$ this proves the required concavity assertion for the source set $\mathcal{M}_\varphi$.

For the level set $\mathcal{E}_\psi$ the proof is similar. We start from

$$\tau(A, \mathcal{E}_\psi, \delta) = \sup \left\{ \|g\|_*^2 : \int\limits_{0<f(\omega)\le t} |g(\omega)|^2 \; d\mu(\omega) \le \psi^2(t) \right\}.$$

Again we choose h_1, h_2 such that

$$\|h_1\|_*^2 \ge \tau(A, \mathcal{E}_\psi, \delta_1) - \varepsilon \quad \text{and} \quad \|h_2\|_*^2 \ge \tau(A, \mathcal{E}_\psi, \delta_2) - \varepsilon,$$

together with

$$\int\limits_{0<f(\omega)\le t} |h_1(\omega)|^2 \; d\mu(\omega) \le \psi^2(t), \quad 0 < t \le a,$$

and

$$\int\limits_{0<f(\omega)\le t} |h_2(\omega)|^2 \; d\mu(\omega) \le \psi^2(t), \quad 0 < t \le a.$$

The same choice of h as above allows us to complete the proof.

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Rates for smoothness classes

Proposition (Rates for lower and upper bounds)

Let ψ be an index function with $\Theta(t) := \sqrt{t}\psi(t),\ 0 < t \le a$. Then

$$\omega\left(A, \mathcal{E}_{\psi,E}, \delta\right) \le 2\,E\,\psi(\Theta^{-1}(\delta/E)), \quad 0 < \delta \le E\,\Theta(a),$$

and

$$\omega\left(A, \mathcal{E}_{\psi,E}, \delta\right) \ge E\,\psi(\Theta^{-1}(\delta/E)), \quad \delta^2/E^2 \in \sigma(H\psi^2(H)).$$

The upper bound is proven by using **spectral cut-off**.

The lower bound follows from the implication

$$\mathcal{M}_{\psi,E} \subset \mathcal{E}_{\psi,E} \qquad \Longrightarrow \qquad \omega\left(A, \mathcal{E}_{\psi,E}, \delta\right) \ge \omega\left(A, \mathcal{M}_{\psi,E}, \delta\right),$$

and $\omega\left(A, \mathcal{M}_{\psi,E}, \delta\right) \ge E\,\psi(\Theta^{-1}(\delta/E))$ for $\delta^2/E^2 \in \sigma(H\psi^2(H))$.