Ergänzungen I zur Vorlesung Inverse Probleme Sommersem. 2018

On Nashed's ill-posedness concept

BERND HOFMANN

TECHNISCHE UNIVERSITÄT CHEMNITZ

TU Chemnitz
Fakultät für Mathematik
D-09107 Chemnitz, GERMANY

Mathematik!

Mai 2018

Email: hofmannb@mathematik.tu-chemnitz.de

Internet: http://www.tu-chemnitz.de/mathematik/ip/

Outline

1. Nashed's ill-posedness concept for injective operators
2. Two papers on strictly singular operators
3. Distinction of well-posed and ill-posed situations

Nashed's ill-posedness concept for injective operators

The definition follows the suggestions of M. Z. NASHED (1987):

Definition

Let $B : Z_1 \to Z_2$ be an **injective and bounded linear operator** mapping between the infinite dimensional Banach spaces Z_1 and Z_2. Then the operator equation

$$Bx = y\,, \quad x \in Z_1,\; y \in \mathcal{R}(B) \subseteq Z_2 \qquad (\$)$$

is called **well-posed** if the range $\mathcal{R}(B)$ is a closed subset of Z_2, consequently **ill-posed** if the range is not closed. In the ill-posed case, the equation (\$) is called **ill-posed of type I** if the range $\mathcal{R}(B)$ contains an **infinite dimensional closed subspace**, and it is called **ill-posed of type II** otherwise.

Roughly speaking:

$B_1 : Z_1 \to Z_2$ with ill-posedness of type I is

'more smoothing' than

$B_2 : Z_1 \to Z_2$ with ill-posedness of type II

if our focus is on **range inclusions**. Precisely:

$\mathcal{R}(B_2) \subset \mathcal{R}(B_1)$ may occur, but $\mathcal{R}(B_1) \subset \mathcal{R}(B_2)$ cannot occur.

Example ($\triangleright$ BÖTTCHER/H./TAUTENHAHN/YAMAMOTO 2006) $\quad Z_1 = Z_2 = L^2(0,1)$

$$[B_1 x](t) = t^\kappa \, x(t) \quad \text{noncompact multiplication operator,}$$

$$B_2 : \; \mathcal{R}(B_2) = H^r(0,1) \; (0 \le r \le 1) \quad \text{compact integration operator,}$$

$$\text{and we have} \quad \mathcal{R}(B_2) \subset L^\infty(0,1) \subset \mathcal{R}(B_1) \quad \text{if} \;\; 0 < \kappa < 1/2.$$

Proposition

If the operator equation ($) is well-posed or ill-posed of type I, then the operator B is non-compact. If, on the other hand, B is compact, then ($) is ill-posed of type II. In particular, if Z_1 and Z_2 in equation ($) are Hilbert spaces and the equation is ill-posed, then we have ill-posedness of type II if the operator B is compact and ill-posedness of type I if B is not compact.

Proof: If ($) is well-posed or ill-posed of type I, there is an infinite dimensional Banach space $\hat{Z}_2$ included in the subspace $\mathcal{R}(B)$ of Z_2 with the same norm as in Z_2. $\hat{Z}_1 := B^{-1}(\hat{Z}_2)$ is a Banach space included in Z_1 with the same norm as in Z_1. For compact B the restriction $B|_{\hat{Z}_1} : \hat{Z}_1 \to \hat{Z}_2$ would be compact and moreover surjective. This contradicts the fact that a compact operator has only a closed range if the range is finite dimensional. For Hilbert spaces Z_1 and Z_2 and ill-posed ($), equivalence of ill-posedness of type I and non-compactness of B is well-known (cf. NASHED (1986) or DOUGLAS (1998)).

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Two papers on strictly singular operators

Definition

Let $B : Z_1 \to Z_2$ be a bounded linear operator mapping between the infinite dimensional Banach spaces Z_1 and Z_2. Then B is said to be **strictly singular** if the restriction of B to an infinite dimensional subspace of Z_1 is never an isomorphism (i.e. never a linear homeomorphism).

$\triangleright$ S. Goldberg and E. O. Thorp: On some open questions concerning strictly singular operators. Proc. Amer. Math. Soc., 14:334–336, 1963.

$\triangleright$ E. Tarafdar: A note on the bounded linear operators on the spaces ℓ^p and L_p. J. Math. Anal. Appl., 40:683–686, 1972.

From the history:

- Every compact operator $B : Z_1 \to Z_2$ is strictly singular.
- In Hilbert spaces Z_1, Z_2 strictly singular operators are compact.
- T. KATO (1958) analyzed the concept of strictly singular operators B and asked the question:
  Is every strictly singular operator B compact?

Theorem (S. GOLDBERG AND E. O. THORP)

(a) Every bounded linear operator B from ℓ^2 to ℓ^p or from ℓ^p to ℓ^2 for $1 \le p \ne 2, \infty$, is strictly singular.
(b) Let Z_1 be a Banach space which does not contain an infinite dimensional reflexive subspace and Z_2 a reflexive space. Then every bounded linear operator B from Z_1 to Z_2 or from Z_2 to Z_1 is strictly singular.

ad (a): Embedding operators from ℓ^2 to $\ell^p,\ 2 < p < \infty$, and from ℓ^p to $\ell^2,\ 1 \le p < 2$, are strictly singular.
ad (b): See $Z_1 = \ell^1$ or $Z_1 = c_0$!

R. S. PHILLIPS presented in an unpublished note a bounded linear operator B which is strictly singular and not compact, but Theorem above shows that the closed subspace $\mathcal{S}(Z_1, Z_2)$ of strictly singular operators in $\mathcal{L}(Z_1, Z_2)$ is 'much larger' than the corresponding set of all compact operators $\mathcal{C}(Z_1, Z_2)$. Moreover, Phillips considered in his note a concept of bounded linear operators $\mathcal{R}(Z_1, Z_2)$ whose range does not contain any infinite dimensional closed subspace.

We have $\mathcal{C}(Z_1, Z_2) \subseteq \mathcal{R}(Z_1, Z_2) \subseteq \mathcal{S}(Z_1, Z_2) \subseteq \mathcal{L}(Z_1, Z_2)$.

Assertions (TARAFDAR)

(A) $\mathcal{S}(\ell^p, \ell^q) = \mathcal{L}(\ell^p, \ell^q)$ for $1 \le p, q < \infty,\ p \neq q$.

(B) $\mathcal{R}(\ell^p, \ell^q) = \mathcal{L}(\ell^p, \ell^q)$ for $1 < p < \infty,\ 1 \le q < \infty,\ p \neq q$.

(C) $\mathcal{C}(\ell^p, \ell^q) = \mathcal{L}(\ell^p, \ell^q)$ for $1 \le p, q < \infty,\ q < p$.

Remarks (S. Goldberg and E. O. Thorp)

(i) There is a bounded linear operator B mapping ℓ^1 onto ℓ^2 by Theorem of Banach/Mazur, since ℓ^2 is a separable Banach space. By the theorem this operator is strictly singular and not compact, but unfortunately also not injective.
(ii) The Kato and Phillips concept coincide if the domain space Z_1 is ℓ^2, i.e. $\mathcal{S}(\ell^2, Z_2) = \mathcal{R}(\ell^2, Z_2)$.

Assertions

(a) If Z_1 is a separable Hilbert space, then the bounded linear operator $B: Z_1 \to Z_2$ is strictly singular if and only if its range does not contain any infinite dimensional closed subspace.

Consequently:

(b) For an arbitrary Banach spaces Z_1 and an injective bounded linear operator B the operator equation (\$) is always ill-posed of type II if B is strictly singular.

(c) For $Z_1 = \ell^1$, a reflexive Banach space Z_2 and an injective bounded linear operator B the operator equation (\$) is always ill-posed of type II.

Outline

Distinction of well-posed and ill-posed situations

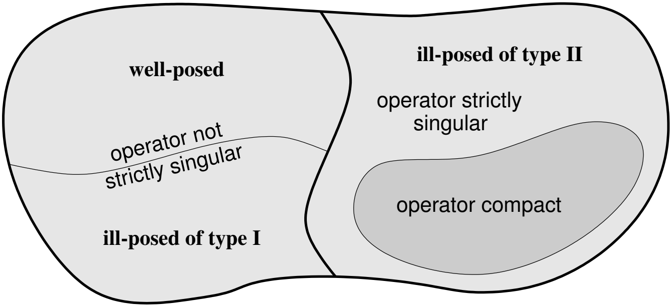

Figure: Well-posedness and ill-posedness types of equation ($)