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Lavrentiev regularization in Hilbert space

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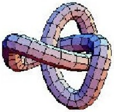

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Relevant and recently published or submitted papers:

▷ RADU I. BOŢ, B. HOFMANN: Conditional stability versus ill-posedness for operator equations with monotone operators in Hilbert space. *Inverse Problems* **32**, No. 12 (2016), 125003 (23pp).

▷ B. HOFMANN, B. KALTENBACHER, E. RESMERITA: Lavrentiev's regularization method in Hilbert spaces revisited. *Inverse Probl. Imaging* **10**, No. 3 (2016), pp. 741–764.

▷ R. PLATO, P. MATHÉ, B. HOFMANN: Optimal rates for Lavrentiev regularization with adjoint source conditions. Paper submitted 2016. Preprint 2016-3, Technische Universität Chemnitz, Fakultät für Mathematik.

▷ R. PLATO: Converse results, saturation and quasi-optimality for Lavrentiev regularization of accretive problems. Paper submitted 2016. Preliminary version published electronically under arXiv:1607.04879v1.

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1. Introduction
2. Assumptions and basics for Lavrentiev regularization
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Let X denote a **separable real Hilbert space** with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$.

For a **monotone operator** $F:\mathcal{D}(F)\subseteq X\to X,$ i.e.,

$$\langle F(x)-F(\tilde{x}),x-\tilde{x}\rangle\geq 0 \qquad \text{for all} \quad x,\tilde{x}\in\mathcal{D}(F), \qquad (Mon)$$

we consider the (possibly nonlinear) operator equation

$$\boxed{F(x)\;=\;y \qquad (x\in\mathcal{D}(F)\subseteq X,\;y\in X) \qquad (**)}$$

with solution $x^\dagger\in\mathcal{D}(F)$ and exact right-hand side $y=F(x^\dagger)$.

The **goal** is to find approximations to $x^\dagger$ with good properties based on **noisy data** $y^\delta \in X$ such that

$$\|y - y^\delta\| \le \delta, \qquad\qquad (\textit{Noise})$$

with noise level $\delta > 0$.

If solving equation $(**)$ is the model for an **inverse problem**, i.e., the **forward operator** F is '**smoothing**', then a least squares approach

$$\|F(x) - y^\delta\|^2 \to \min, \quad \text{subject to} \quad x \in \mathcal{D}(F).$$

is not always successful, even if $x^\dagger$ is the unique solution to $(**)$.

We recall the following concept from ▷ H./SCHERZER IP 1994.

Definition

The equation $(**)$ is called **locally well-posed** at the solution point $x^\dagger \in \mathcal{D}(F)$ if there is a ball $B_r(x^\dagger)$ with radius $r > 0$ and center $x^\dagger$ such that for each sequence $\{x_k\}_{k=1}^\infty \subset B_r(x^\dagger) \cap \mathcal{D}(F)$ the convergence of images $\lim\limits_{k\to\infty} \|F(x_k) - F(x^\dagger)\| = 0$ implies the convergence of the preimages $\lim\limits_{k\to\infty} \|x_k - x^\dagger\| = 0$.
Otherwise it is called **locally ill-posed** at $x^\dagger$.

If $(**)$ is a model of an inverse problem, then due to local ill-posedness it makes sense to exploit a **singularly perturbed auxiliary problem** to equation $(**)$, which in general proves to be locally well-posed.

Singularly perturbed auxiliary problems

In addition to the most popular **Tikhonov regularization** for **general 'smoothing'** forward operators F, where regularized solutions x_α^δ are minimizers of

$$\|F(x)-y^\delta\|^2+\|x-\bar{x}\|^2 \to \min, \qquad \text{subject to} \quad x \in \mathcal{D}(F)\,, \qquad (Tik)$$

we have the simpler **Lavrentiev regularization** for **monotone** operators F, where x_α^δ solves the singularly perturbed operator equation

$$F(x) + \alpha(x - \bar{x}) = y^\delta\,. \qquad (Lav)$$

Conditional stability

Instability arising from ill-posedness can also be overcome by having **conditional stability estimates** of the form

$$\|x - x^\dagger\| \le \varphi(\|F(x) - F(x^\dagger)\|) \quad \text{for all} \quad x \in \mathcal{D}(F) \cap Q \qquad (CSE)$$

for an **index function** φ and an appropriate set $Q \supset \{x^\dagger\}$, where $\varphi : [0,\infty) \to [0,\infty)$ is an index function if it is continuous and strictly increasing with $\varphi(0) = 0$.

Often Q depends on properties of $x^\dagger$ and is not known a priori. Consequently, (CSE) is not directly applicable for finding stable approximate solutions to $(**)$. Thus, additional tools are needed.

▷ CHENG/YAMAMOTO IP 2000 ▷ H./YAMAMOTO IP 2010

Verification of CSE-tools for φ concave and $Q = B_r(x^\dagger)$

Under $\|x - x^\dagger\| \le \varphi(\|F(x) - F(x^\dagger)\|)$ for all $x \in \mathcal{D}(F) \cap B_r(x^\dagger)$ solution $x^\dagger$ is **unique** in $B_r(x^\dagger)$, $(**)$ is **locally well-posed** at $x^\dagger$.

Tikhonov regularization for general forward operators F:

$$x^\delta_{\alpha(\delta)} = \arg\min_{x \in \mathcal{D}(F)} \left\{ \|F(x) - y^\delta\|^2 + \|x - \bar{x}\|^2 \right\} \text{ for } \alpha(\delta) = c\,\delta^2$$

yields with $\|x^\delta_{\alpha(\delta)} - \bar{x}\|^2 \le \frac{\delta^2}{\alpha} + \|x^\dagger - \bar{x}\|^2$ and triangle inequality $\|x^\delta_{\alpha(\delta)} - x^\dagger\| \le \sqrt{\frac{\delta^2}{\alpha}} + 2\|x^\dagger - \bar{x}\| = \sqrt{\frac{1}{c}} + 2\|x^\dagger - \bar{x}\|$ as well as $\|F(x^\delta_{\alpha(\delta)}) - F(x^\dagger)\| \le \left(2 + \sqrt{c}\,\|x^\dagger - \bar{x}\|\right)\delta$. For $r > \sqrt{\frac{1}{c}} + 2\|x^\dagger - \bar{x}\|$ we arrive at (CSE) $\|x^\delta_{\alpha(\delta)} - x^\dagger\| \le \varphi(\|F(x^\delta_{\alpha(\delta)}) - F(x^\dagger)\|)$. Hence:

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| \le \left(2 + \sqrt{c}\,\|x^\dagger - \bar{x}\|\right)\,\varphi(\delta).$$

Verification of CSE-tools for φ concave and $Q = B_r(x^\dagger)$

Lavrentiev regularization for monotone F and $\mathcal{D}(F) = X$:

$$F(x^\delta_{\alpha(\delta)}) + \alpha(x^\delta_{\alpha(\delta)} - \bar{x}) = y^\delta \qquad \text{for} \quad \alpha(\delta) = c\,\delta$$

yields with $\|x^\delta_{\alpha(\delta)} - \bar{x}\| \le \frac{\delta}{\alpha} + \|x^\dagger - \bar{x}\|$ and

$$\|F(x^\delta_{\alpha(\delta)}) - F(x^\dagger)\| \le \alpha\,\|x^\dagger - \bar{x}\| + \delta = (c\,\|x^\dagger - \bar{x}\| + 1)\,\delta.$$

For $r > \frac{1}{c} + \|x^\dagger - \bar{x}\|$ we arrive at

$$(CSE) \quad \|x^\delta_{\alpha(\delta)} - x^\dagger\| \le \varphi(\|F(x^\delta_{\alpha(\delta)}) - F(x^\dagger)\|), \quad \text{which implies}$$

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| \le 2\max(1, c\,\|x^\dagger - \bar{x}\|)\,\varphi(\delta).$$

Using regularization under conditional stability is like putting into the hole Q while playing golf!

Outline

Standing assumptions

- $F : X \to X,\ \mathcal{D}(F) = X$ (X separable real Hilbert space).
- F is a monotone and hemicontinuous operator.

Then F is even **maximally monotone** and we have a weak-to-norm sequential closedness as

$$x_n \rightharpoonup \tilde{x} \quad \text{and} \quad F(x_n) \to z_0 \quad \Rightarrow \quad F(\tilde{x}) = z_0.$$

Under the standing assumptions there occur well-posed and ill-posed situations. The best situation of global well-posedness is characterized by **strong monotonicity**

$$\langle F(x) - F(\tilde{x}), x - \tilde{x} \rangle \ge C \, \|x - \tilde{x}\|^2 \qquad \text{for all} \quad x, \tilde{x} \in X,$$

with some constant $C > 0$, which implies the **coercivity** condition

$$\lim_{\|x\| \to \infty} \frac{\langle F(x), x \rangle}{\|x\|} = \infty.$$

Under the standing assumptions $F : X \to X$ is **surjective** due to the **Browder-Minty theorem** if the coercivity condition holds. If, moreover, F is strongly monotone, then F is **bijective** and $F^{-1} : X \to X$ is Lipschitz continuous as

$$\|F^{-1}(y) - F^{-1}(\tilde{y})\| \le \frac{1}{C} \|y - \tilde{y}\| \qquad \text{for all} \quad y, \tilde{y} \in X. \qquad (Lip)$$

There are classes of ill-posed inverse problems with monotone F occurring in natural sciences and engineering, where (Lip) fails. Then we have operator equations $(**)$ of the **first kind**, but the associated equations of the **second kind**

$$G(x) = y \qquad \text{with} \qquad G(x) := F(x) + \alpha x$$

satisfy (Lip) with $C = \alpha$ for all $\alpha > 0$.

This motivates Lavrentiev regularization (Lav) for stabilizing $(**)$.

Example (▷ B. KALTENBACHER)

As an ill-posed example we consider the identification of the source term q in the elliptic boundary value problem

$$-\Delta u + \xi(u) = q \text{ in } \mathcal{G}$$
$$u = 0 \text{ on } \partial\mathcal{G}$$

from measurements of u in $\mathcal{G}$, where $\xi : \mathbb{R} \to \mathbb{R}$ is some Lipschitz continuously differentiable monotonically increasing function and $\mathcal{G} \subseteq \mathbb{R}^3$ a smooth domain.

Then the corresponding **nonlinear** forward operator
$F : X := L^2(\mathcal{G}) \to H^2(\mathcal{G}) \subseteq L^2(\mathcal{G})$,
mapping $q \mapsto u$, is **monotone and hemicontinuous.**

Proposition

Under our assumptions let, for given $y \in X$, the solution set

$$L := \{x \in X : \; F(x) = y\}$$

to equation $(**)$ be nonempty. Then L is closed and convex, and consequently there is a uniquely determined $\bar{x}$-minimum norm solution $x_{mn}^{\dagger} \in L$ to $(**)$ such that

$$\|x_{mn}^{\dagger} - \bar{x}\| = \min\{\|x^{\dagger} - \bar{x}\| : \; x^{\dagger} \in L\}.$$

Moreover, the Lavrentiev-regularized solution $x_{\alpha}^{\delta} \in X$ is uniquely determined, which means that

$$F(x_{\alpha}^{\delta}) + \alpha(x_{\alpha}^{\delta} - \bar{x}) = y^{\delta} \qquad (Lav)$$

has a unique solution x_{α}^{δ} for all $\bar{x} \in X$, $y^{\delta} \in X$ and $\alpha > 0$, where x_{α}^{δ} depends continuously on y^{δ}.

For any $x^\dagger \in L$ the following three basic inequalities are valid:

$$\|x_\alpha^\delta - x^\dagger\|^2 \le \langle x^\dagger - \bar{x}, x^\dagger - x_\alpha^\delta \rangle + \frac{\delta}{\alpha}\|x_\alpha^\delta - x^\dagger\|\,,$$
$$\|x_\alpha^\delta - x^\dagger\| \le \|x^\dagger - \bar{x}\| + \frac{\delta}{\alpha}\,,$$
$$\|F(x_\alpha^\delta) - F(x^\dagger)\| \le \alpha\|x^\dagger - \bar{x}\| + \delta\,.$$

Lavrentiev regularization is always helpful if bijectivity of F and hence Lipschitz property of F^{-1} fails, for example because coercivity fails or well-posedness occurs only in a local sense.

This is the case if F is **locally strongly monotone**

$$\langle F(x) - F(x^\dagger), x - x^\dagger \rangle \geq C\,\|x - x^\dagger\|^2 \qquad \text{for all} \quad x \in B_r(x^\dagger),$$

with $C > 0$ and $r > 0$, or if F is **locally uniformly monotone**

$$\langle F(x) - F(x^\dagger), x - x^\dagger \rangle \geq \zeta(\|x - x^\dagger\|) \qquad \text{for all} \quad x \in B_r(x^\dagger)$$

with some index function ζ and $r > 0$.

Proposition

Let local uniform monotonicity of F in $B_r(x^\dagger)$ hold with an index function ζ of the form $\zeta(t) = \theta(t)\,t,\ t > 0$, such that θ is also an index function. Then we have a conditional stability estimate

$$\|x - x^\dagger\| \le \theta^{-1}(\|F(x) - F(x^\dagger)\|) \quad \text{for all} \quad x \in B_r(x^\dagger), \qquad (CSE)$$

and the operator equation $(**)$ is locally well-posed at the solution point $x^\dagger$.

In the special case $\zeta(t) = C\,t^2$ of strong monotonicity we find

$$\|x - x^\dagger\| \le \frac{1}{C}\,\|F(x) - F(x^\dagger)\| \quad \text{for all} \quad x \in B_r(x^\dagger). \qquad (CSE)$$

One dimensional example

For $X := \mathbb{R}$ with $\|x\| := |x|$ we consider $(**)$ with the continuous monotone operator $F : \mathbb{R} \to \mathbb{R}$ defined for exponents $\kappa > 0$ as

$$F(x) := \begin{cases} -1 & \text{if} \quad -\infty < x < -1 \\ -(-x)^\kappa & \text{if} \quad -1 \le x \le 0 \\ x^\kappa & \text{if} \quad 0 < x \le 1 \\ 1 & \text{if} \quad 1 < x < \infty \end{cases},$$

which, however, is not bijective and not coercive.

Then we have local ill-posedness at $x^\dagger$ if $x^\dagger \le -1$ or $x^\dagger \ge 1$. At $x^\dagger = 0$ we have local well-posedness for all $\kappa > 0$ due to a local uniform monotonicity condition with $\zeta(t) = t^{\kappa+1}$ such that

$$\|x - x^\dagger\| \le \|F(x) - F(x^\dagger)\|^{1/\kappa} \quad \text{for all} \quad x \in B_1(0). \qquad (CSE)$$

Lavrentiev regularization allows for linear convergence if $\kappa = 1$, Hölder convergence rates for $\kappa > 1$, and there even occurs a superlinear convergence rate at $x^\dagger = 0$ if $0 < \kappa < 1$.

Outline

If $(**)$ is **locally ill-posed** at $x^\dagger$, then $\alpha(\delta) \sim \delta$ is not appropriate for Lavrentiev regularization, and the decay of $\alpha(\delta)$ as $\delta \to 0$ has to be slower in order to ensure convergence.

We have for the **bias** $\|x_\alpha - x^\dagger\|$ with $F(x_\alpha) + \alpha(x_\alpha - \bar{x}) = y$:

Proposition

$$\lim_{\alpha \to 0} \|x_\alpha - x^\dagger\| = 0 \qquad \text{if and only if} \qquad x^\dagger = x^\dagger_{mn},$$

$$\|x_\alpha^\delta - x^\dagger\| \le \|x_\alpha - x^\dagger\| + \frac{\delta}{\alpha}.$$

Consequently, an a priori choice $\alpha(\delta)$ satisfying

$$\alpha(\delta) \to 0 \quad \text{and} \quad \frac{\delta}{\alpha(\delta)} \to 0 \quad \text{as} \quad \delta \to 0$$

yields **convergence** $\lim_{\delta \to 0} \|x^\delta_{\alpha(\delta)} - x^\dagger_{mn}\| = 0$.

Under nonlinearity conditions on F Hölder **convergence rates**

$$\|x_\alpha^\delta - x^\dagger\| = O\left(\delta^{\frac{p}{p+1}}\right) \qquad \text{as} \qquad \delta \to 0,$$

occur for $0 < p \le 1$ with $A := F'(x^\dagger)$ and $\alpha(\delta) \sim \delta^{\frac{1}{p+1}}$ with the power-type source condition

$$x^\dagger - \bar{x} = A^p\, w, \qquad w \in X,$$

where the fractional powers A^p of the monotone linear operator A are defined by a Dunford integral as

$$A^p v := \frac{\sin(p\pi)}{\pi} \int_0^\infty s^{p-1}(A + sI)^{-1} Av\, ds, \qquad v \in X.$$

See, e.g., $\triangleright$ TAUTENHAHN 2002.

Let us consider a variational source condition adapted to the monotonicity structure:

Variational source condition (VSC-Lav)

We assume to have a constant $0 \le \beta < 1$ *and an index function* φ *such that for all* $x \in \mathcal{M}$

$$\langle x^\dagger - \bar{x}, x^\dagger - x \rangle \le \beta \|x^\dagger - x\|^2 + \varphi(\langle F(x) - F(x^\dagger), x - x^\dagger \rangle).$$

Here, φ must be an index function with $\lim_{t \to +0} \frac{\varphi(t)}{\sqrt{t}} \ge \underline{c} > 0$.

The next theorem is proved by the general Young inequality

$$ab \le \int_0^a f(t)\,dt + \int_0^b f^{-1}(t)\,dt$$

for $a, b \ge 0$ and an index function f.

Theorem

Let under the standing assumptions ($VSC - Lav$) be satisfied for the $\bar{x}$-minimum-norm solution $x^\dagger = x^\dagger_{mn}$ of $(**)$ with $\mathcal{M}$ such that for the choice of $\alpha > 0$ all regularized solutions x_α^δ from (Lav) belong to $\mathcal{M}$ for sufficiently small $\delta > 0$. Then one has the estimate

$$\|x_\alpha^\delta - x^\dagger\|^2 \leq \frac{1}{(1-\beta)^2}\frac{\delta^2}{\alpha^2} + \frac{2}{1-\beta}\Psi(\alpha),$$

for such $\delta > 0$, where $\Psi(\alpha)$ is introduced as follows:
Let f be an index function such that its antiderivative $\tilde{f}(s) := \int_0^s f(t)dt$ satisfies the condition $\tilde{f}(\varphi(s)) \leq s$ for $s > 0$ and let $G(\alpha) \geq \int_0^\alpha f^{-1}(\tau)d\tau$. Then we set $\Psi(\alpha) := \frac{G(\alpha)}{\alpha}$.

Theorem cont.

Moreover, for the a priori choice

$$\alpha(\delta) \sim \Theta^{-1}(\delta^2)$$

with $\Theta(\alpha) := \alpha^2 \Psi(\alpha)$, which satisfies

$$\alpha(\delta) \to 0 \quad \text{and} \quad \frac{\delta}{\alpha(\delta)} \to 0 \quad \text{as} \quad \delta \to 0,$$

this yields the convergence rate

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| = O\left(\frac{\delta}{\Theta^{-1}(\delta^2)}\right) = O\left(\sqrt{\Psi(\Theta^{-1}(\delta^2))}\right).$$

The best possible rate occurs for $\varphi(t) \sim \sqrt{t}$ and $\Psi(t) \sim t$ as

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| = O(\delta^{\frac{1}{3}}) \qquad \text{if} \quad \alpha(\delta) \sim \delta^{\frac{2}{3}}.$$

Logarithmic type variational source conditions

We consider $(VSC - Lav)$ with

$$\varphi(t) = \frac{1}{-\ln t}, \quad f(t) = e^{-\frac{1}{t}} \le \frac{1}{t^2} e^{-\frac{1}{t}} = \varphi^{-1\prime}(t), \quad f^{-1}(t) = \frac{1}{-\ln t},$$

$$G(\alpha) = \alpha \frac{1}{-\ln \alpha} \ge \int_0^\alpha \frac{1}{-\ln t}\, dt, \quad \Theta(t) \sim t^2 \frac{1}{-\ln t}, \text{ for } \alpha, t \in (0,1).$$

This yields the logarithmic rate $O\left(\sqrt{\Psi(\Theta^{-1}(\delta^2))}\right)$.

Since the a priori choice $\alpha(\delta) \sim \Theta^{-1}(\delta^2)$ cannot be determined explicitly in this logarithmic case, a more convenient choice is $\alpha(\delta) \sim \sqrt{\delta}$ which implies

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| = O\left(\frac{1}{\sqrt{-\ln(\delta)}}\right) .$$

Hölder type variational source conditions

For exponents $\mu \in (0, \frac{1}{2}]$, we consider $(VSC - Lav)$ with

$$\varphi(t) = t^\mu, \quad f(t) = \frac{1}{\mu} t^{\frac{1-\mu}{\mu}}, \quad f^{-1}(t) = (\mu t)^{\frac{\mu}{1-\mu}},$$

$$\tilde{f}(s) = s^{\frac{1}{\mu}}, \quad G(\alpha) = (1-\mu)\mu^{\frac{\mu}{1-\mu}} \alpha^{\frac{1}{1-\mu}}, \quad \Phi(t) \sim t^{\frac{2-\mu}{1-\mu}}.$$

According to the theorem this yields the Hölder rate

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| = O(\delta^{\frac{\mu}{2-\mu}}) \qquad \text{if} \quad \alpha(\delta) \sim \delta^{\frac{2(1-\mu)}{2-\mu}}.$$

Note that $O\left(\delta^{\frac{\mu}{2-\mu}}\right) = O\left(\delta^{\frac{p}{p+1}}\right)$ for $\mu := \frac{2p}{2p+1}$, $0 < p \le \frac{1}{2}$.

For linear forward operators $F := A \in \mathcal{L}(X, X)$ we outline situations of relations to the variational source conditions:

Selfadjoint A: First let $A = A^*$, $x^\dagger = A^p w$, $\bar{x} := 0$ and $0 < p \le \frac{1}{2}$.

$$\begin{aligned} \langle x^\dagger, x \rangle &= \langle w, A^p x \rangle \le \|w\| \, \|A^p x\| \\ &\le \|w\| \, \|x\|^{1-2p} \|A^{\frac{1}{2}} x\|^{2p} = \|w\| \, \|x\|^{1-2p} \langle Ax, x \rangle^p \end{aligned}$$

based on the interpolation inequality. By using Young's ineq. $ab \le \frac{a^\xi}{\xi} + \frac{b^\eta}{\eta}$ with $a = \|x\|^{1-2p}$, $b = \|w\| \, \langle Ax, x \rangle^p$, $\xi = \frac{2}{1-2p}$ and $\eta = \frac{2}{1+2p}$ this yields the variational source condition

$$\langle x^\dagger, x \rangle \le \left(\frac{1}{2} - p \right) \|x\|^2 + \left(\frac{1}{2} + p \right) \|w\|^{\frac{2}{2p+1}} \, \langle Ax, x \rangle^{\frac{2p}{2p+1}} \quad \forall x \in X$$

with exponent $0 < \mu = \frac{2p}{2p+1} \le \frac{1}{2}$, hence the Hölder rate

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| = O(\delta^{\frac{\mu}{2-\mu}}) = O(\delta^{\frac{p}{p+1}}) \qquad \text{if} \quad \alpha(\delta) \sim \delta^{\frac{2(1-\mu)}{2-\mu}} = \delta^{\frac{1}{p+1}}.$$

Non-selfadjoint A: Secondly, we consider for $X = L^2(0,1)$, the linear integration operator

$$[Ax](s) := \int_0^s x(t)\,dt, \quad 0 \le s \le 1,$$

which is monotone, but not selfadjoint, i.e. $A \neq A^*$. We have

$$\langle Ax, x\rangle = \frac{1}{2}\left(\int_0^1 x(t)\,dt\right)^2 \ge 0,$$

and for $x^\dagger \equiv 1$, with $x^\dagger \notin \mathcal{R}(A)$ and $x^\dagger \notin \mathcal{R}(A^*)$,

$$\langle x^\dagger, x\rangle = \int_0^1 x(t)\,dt \le \sqrt{2}\langle Ax, x\rangle^{\frac{1}{2}} \qquad \forall x \in X,$$

thus the convergence rate for the Lavrentiev regularization

$$\|x_\alpha^\delta - x^\dagger\| = O\left(\delta^{\frac{1}{3}}\right) \quad \text{for } \alpha \sim \delta^{\frac{2}{3}}.$$

Outline

Lavrentiev regularization for linear operator equations

Let $A : X \to X$ be a bounded linear monotone operator. For

$$Ax = y \qquad (*)$$

the linear Lavrentiev regularization acts as

$$x_\alpha^\delta = (A + \alpha I)^{-1}(y^\delta + \alpha \bar{x}).$$

Proposition

The linear equation $(*)$ is **locally well-posed everywhere** iff A is continuously invertible, i.e. there is $K > 0$ such that

$$\|(A + \alpha I)^{-1}\| \le K < \infty \quad \text{for all} \quad \alpha > 0,$$

where $K = \|A^{-1}\|$ holds true, but $(*)$ is **locally ill-posed everywhere** iff $\mathcal{N}(A) \ne \{0\}$ or $\mathcal{R}(A) \ne \overline{\mathcal{R}(A)}$. Then we have

$$\|(A + \alpha I)^{-1}\| = 1/\alpha \quad \text{for all} \quad \alpha > 0.$$

Theorem (▷ Plato 2016)

For the maximal best possible error

$$E_{x^\dagger}(\delta) := \sup_{y^\delta \in X:\, \|y-y^\delta\| \le \delta} \; \inf_{\alpha>0} \|x_\alpha^\delta - x^\dagger\|$$

of Lavrentiev regularization to the linear equation $(*)$ we have

$$E_{x^\dagger}(\delta) = \mathcal{O}(\delta) \quad \text{as} \quad \delta \to 0$$

in the well-posed case, but

$$E_{x^\dagger}(\delta) = o(\sqrt{\delta}) \quad \text{as} \quad \delta \to 0 \quad \text{implies} \quad x^\dagger - \bar{x} = 0$$

in the ill-posed case.