Lavrentiev regularization in Hilbert space

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4. Lavrentiev regularization for linear operator equations
Let $X$ denote a **separable real Hilbert space** with norm $\| \cdot \|$ and inner product $\langle \cdot, \cdot \rangle$.

For a **monotone operator** $F : \mathcal{D}(F) \subseteq X \rightarrow X$, i.e.,

$$\langle F(x) - F(\tilde{x}), x - \tilde{x} \rangle \geq 0 \quad \text{for all } x, \tilde{x} \in \mathcal{D}(F), \quad (\text{Mon})$$

we consider the (possibly nonlinear) operator equation

$$F(x) = y \quad (x \in \mathcal{D}(F) \subseteq X, \ y \in X) \quad (***)$$

with solution $x^\dagger \in \mathcal{D}(F)$ and exact right-hand side $y = F(x^\dagger)$. 
The **goal** is to find approximations to $x^\dagger$ with good properties based on **noisy data** $y^\delta \in X$ such that

$$\| y - y^\delta \| \leq \delta,$$

(Noise)

with noise level $\delta > 0$.

If solving equation (**) is the model for an **inverse problem**, i.e., the **forward operator** $F$ is ‘smoothing’, then a least squares approach

$$\| F(x) - y^\delta \|^2 \rightarrow \min, \quad \text{subject to} \quad x \in \mathcal{D}(F).$$

is not always successful, even if $x^\dagger$ is the unique solution to (**).
We recall the following concept from H./SCHERZER IP 1994.

**Definition**

The equation (**) is called **locally well-posed** at the solution point \( x^\dagger \in \mathcal{D}(F) \) if there is a ball \( B_r(x^\dagger) \) with radius \( r > 0 \) and center \( x^\dagger \) such that for each sequence \( \{x_k\}_{k=1}^{\infty} \subset B_r(x^\dagger) \cap \mathcal{D}(F) \) the convergence of images \( \lim_{k \to \infty} \| F(x_k) - F(x^\dagger) \| = 0 \) implies the convergence of the preimages \( \lim_{k \to \infty} \| x_k - x^\dagger \| = 0 \).

Otherwise it is called **locally ill-posed** at \( x^\dagger \).

If (**) is a model of an inverse problem, then due to local ill-posedness it makes sense to exploit a **singly perturbed auxiliary problem** to equation (**), which in general proves to be locally well-posed.
Singularly perturbed auxiliary problems

In addition to the most popular Tikhonov regularization for general ‘smoothing’ forward operators $F$, where regularized solutions $x_\alpha^\delta$ are minimizers of

$$\|F(x) - y_\delta\|^2 + \|x - \bar{x}\|^2 \to \text{min}, \quad \text{subject to } x \in \mathcal{D}(F), \quad (Tik)$$

we have the simpler Lavrentiev regularization for monotone operators $F$, where $x_\alpha^\delta$ solves the singularly perturbed operator equation

$$F(x) + \alpha(x - \bar{x}) = y_\delta. \quad (Lav)$$
Instability arising from ill-posedness can also be overcome by having **conditional stability estimates** of the form

\[
\| x - x^\dagger \| \leq \varphi(\| F(x) - F(x^\dagger) \|) \quad \text{for all} \quad x \in \mathcal{D}(F) \cap Q \quad (CSE)
\]

for an **index function** \( \varphi \) and an appropriate set \( Q \supset \{ x^\dagger \} \), where \( \varphi : [0, \infty) \to [0, \infty) \) is an index function if it is continuous and strictly increasing with \( \varphi(0) = 0 \).

Often \( Q \) depends on properties of \( x^\dagger \) and is not known a priori. Consequently, \((CSE)\) is not directly applicable for finding stable approximate solutions to \((**\))\). Thus, additional tools are needed.

▷ CHENG/YAMAMOTO IP 2000    ▷ H./YAMAMOTO IP 2010
Verification of CSE-tools for $\varphi$ concave and $Q = B_r(x^\dagger)$

Under $\|x - x^\dagger\| \leq \varphi(\|F(x) - F(x^\dagger)\|)$ for all $x \in \mathcal{D}(F) \cap B_r(x^\dagger)$ solution $x^\dagger$ is unique in $B_r(x^\dagger)$, (***) is locally well-posed at $x^\dagger$.

**Tikhonov regularization for general forward operators $F$:**

\[
x_{\alpha(\delta)}^{\delta} = \arg\min_{x \in \mathcal{D}(F)} \{ \|F(x) - y^\delta\|^2 + \|x - \bar{x}\|^2 \} \quad \text{for} \quad \alpha(\delta) = c \delta^2
\]

yields with $\|x_{\alpha(\delta)}^{\delta} - \bar{x}\|^2 \leq \frac{\delta^2}{\alpha} + \|x^\dagger - \bar{x}\|^2$ and triangle inequality

\[
\|x_{\alpha(\delta)}^{\delta} - x^\dagger\| \leq \sqrt{\frac{\delta^2}{\alpha}} + 2\|x^\dagger - \bar{x}\| = \sqrt{\frac{1}{c}} + 2\|x^\dagger - \bar{x}\| \quad \text{as well as}
\]

\[
\|F(x_{\alpha(\delta)}^{\delta}) - F(x^\dagger)\| \leq (2 + \sqrt{c}\|x^\dagger - \bar{x}\|)\delta. \quad \text{For} \quad r > \sqrt{\frac{1}{c}} + 2\|x^\dagger - \bar{x}\|
\]

we arrive at (CSE) $\|x_{\alpha(\delta)}^{\delta} - x^\dagger\| \leq \varphi(\|F(x_{\alpha(\delta)}^{\delta}) - F(x^\dagger)\|)$. Hence:

\[
\|x_{\alpha(\delta)}^{\delta} - x^\dagger\| \leq \left(2 + \sqrt{c}\|x^\dagger - \bar{x}\|\right) \varphi(\delta).
\]
Verification of CSE-tools for $\varphi$ concave and $Q = B_r(x^\dagger)$

**Lavrentiev regularization for monotone $F$ and $\mathcal{D}(F) = X$:**

$$F(x^\delta_{\alpha(\delta)}) + \alpha(x^\delta_{\alpha(\delta)} - \bar{x}) = y^\delta \quad \text{for} \quad \alpha(\delta) = c \delta$$

yields with

$$\|x^\delta_{\alpha(\delta)} - \bar{x}\| \leq \frac{\delta}{\alpha} + \|x^\dagger - \bar{x}\| \quad \text{and} \quad \|F(x^\delta_{\alpha(\delta)}) - F(x^\dagger)\| \leq \alpha\|x^\dagger - \bar{x}\| + \delta = (c\|x^\dagger - \bar{x}\| + 1) \delta.$$  

For $r > \frac{1}{c} + \|x^\dagger - \bar{x}\|$ we arrive at

$$(CSE) \quad \|x^\delta_{\alpha(\delta)} - x^\dagger\| \leq \varphi(\|F(x^\delta_{\alpha(\delta)}) - F(x^\dagger)\|), \quad \text{which implies} \quad \|x^\delta_{\alpha(\delta)} - x^\dagger\| \leq 2 \max(1, c\|x^\dagger - \bar{x}\|) \varphi(\delta).$$

**Using regularization under conditional stability is like putting into the hole Q while playing golf!**
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Standing assumptions

- $F : X \to X$, $\mathcal{D}(F) = X$ (X separable real Hilbert space).
- $F$ is a monotone and hemicontinuous operator.

Then $F$ is even **maximally monotone** and we have a weak-to-norm sequential closedness as

$$x_n \rightharpoonup \tilde{x} \quad \text{and} \quad F(x_n) \to z_0 \quad \Rightarrow \quad F(\tilde{x}) = z_0.$$

Under the standing assumptions there occur well-posed and ill-posed situations. The best situation of global well-posedness is characterized by **strong monotonicity**

$$\langle F(x) - F(\tilde{x}), x - \tilde{x} \rangle \geq C \|x - \tilde{x}\|^2 \quad \text{for all} \quad x, \tilde{x} \in X,$$

with some constant $C > 0$, which implies the **coercivity** condition

$$\lim_{\|x\| \to \infty} \frac{\langle F(x), x \rangle}{\|x\|} = \infty.$$
Under the standing assumptions $F : X \to X$ is **surjective** due to the **Browder-Minty theorem** if the coercivity condition holds. If, moreover, $F$ is strongly monotone, then $F$ is **bijective** and $F^{-1} : X \to X$ is Lipschitz continuous as

$$
\|F^{-1}(y) - F^{-1}(\tilde{y})\| \leq \frac{1}{C} \|y - \tilde{y}\| \quad \text{for all } y, \tilde{y} \in X. \quad \text{(Lip)}
$$

There are classes of ill-posed inverse problems with monotone $F$ occurring in natural sciences and engineering, where (Lip) fails. Then we have operator equations (**) of the **first kind**, but the associated equations of the **second kind**

$$
G(x) = y \quad \text{with} \quad G(x) := F(x) + \alpha x
$$

satisfy (Lip) with $C = \alpha$ for all $\alpha > 0$.

This motivates Lavrentiev regularization (Lav) for stabilizing (**).
As an ill-posed example we consider the identification of the source term $q$ in the elliptic boundary value problem

$$-\Delta u + \xi(u) = q \text{ in } \mathcal{G}$$

$$u = 0 \text{ on } \partial \mathcal{G}$$

from measurements of $u$ in $\mathcal{G}$, where $\xi : \mathbb{R} \to \mathbb{R}$ is some Lipschitz continuously differentiable monotonically increasing function and $\mathcal{G} \subseteq \mathbb{R}^3$ a smooth domain.

Then the corresponding nonlinear forward operator $F : X := L^2(\mathcal{G}) \to H^2(\mathcal{G}) \subseteq L^2(\mathcal{G})$, mapping $q \mapsto u$, is monotone and hemicontinuous.
Proposition

Under our assumptions let, for given \( y \in X \), the solution set

\[
L := \{ x \in X : F(x) = y \}
\]

to equation \((**)\) be nonempty. Then \( L \) is closed and convex, and consequently there is a uniquely determined \( \bar{x} \)-minimum norm solution \( x_{mn}^\dagger \in L \) to \((**)\) such that

\[
\| x_{mn}^\dagger - \bar{x} \| = \min \{ \| x^\dagger - \bar{x} \| : x^\dagger \in L \}. 
\]

Moreover, the Lavrentiev-regularized solution \( x_\alpha^\delta \in X \) is uniquely determined, which means that

\[
F(x_\alpha^\delta) + \alpha(x_\alpha^\delta - \bar{x}) = y_\delta \quad (Lav)
\]

has a unique solution \( x_\alpha^\delta \) for all \( \bar{x} \in X \), \( y_\delta \in X \) and \( \alpha > 0 \), where \( x_\alpha^\delta \) depends continuously on \( y^\delta \).
For any $x^{\dagger} \in L$ the following three basic inequalities are valid:

$$
\|x_\alpha^\delta - x^{\dagger}\|^2 \leq \langle x^{\dagger} - \bar{x}, x^{\dagger} - x_\alpha^\delta \rangle + \frac{\delta}{\alpha} \|x_\alpha^\delta - x^{\dagger}\|
$$

$$
\|x_\alpha^\delta - x^{\dagger}\| \leq \|x^{\dagger} - \bar{x}\| + \frac{\delta}{\alpha}
$$

$$
\|F(x_\alpha^\delta) - F(x^{\dagger})\| \leq \alpha\|x^{\dagger} - \bar{x}\| + \delta.
$$

Lavrentiev regularization is always helpful if bijectivity of $F$ and hence Lipschitz property of $F^{-1}$ fails, for example because coercivity fails or well-posedness occurs only in a local sense.
This is the case if $F$ is \textbf{locally strongly monotone}

\[
\langle F(x) - F(x^\dagger), x - x^\dagger \rangle \geq C \|x - x^\dagger\|^2 \quad \text{for all} \quad x \in B_r(x^\dagger),
\]

with $C > 0$ and $r > 0$, or if $F$ is \textbf{locally uniformly monotone}

\[
\langle F(x) - F(x^\dagger), x - x^\dagger \rangle \geq \zeta(\|x - x^\dagger\|) \quad \text{for all} \quad x \in B_r(x^\dagger)
\]

with some index function $\zeta$ and $r > 0$. 
Proposition

Let local uniform monotonicity of $F$ in $B_r(x^\dagger)$ hold with an index function $\zeta$ of the form $\zeta(t) = \theta(t) t$, $t > 0$, such that $\theta$ is also an index function. Then we have a conditional stability estimate

$$\|x - x^\dagger\| \leq \theta^{-1}(\|F(x) - F(x^\dagger)\|) \quad \text{for all} \quad x \in B_r(x^\dagger), \quad (CSE)$$

and the operator equation (***) is locally well-posed at the solution point $x^\dagger$.

In the special case $\zeta(t) = C t^2$ of strong monotonicity we find

$$\|x - x^\dagger\| \leq \frac{1}{C} \|F(x) - F(x^\dagger)\| \quad \text{for all} \quad x \in B_r(x^\dagger). \quad (CSE)$$
One dimensional example

For \( X := \mathbb{R} \) with \( \| x \| := |x| \) we consider \((**\)) with the continuous monotone operator \( F : \mathbb{R} \rightarrow \mathbb{R} \) defined for exponents \( \kappa > 0 \) as

\[
F(x) := \begin{cases}
-1 & \text{if } -\infty < x < -1 \\
-(1-x)\kappa & \text{if } -1 \leq x \leq 0 \\
x^\kappa & \text{if } 0 < x \leq 1 \\
1 & \text{if } 1 < x < \infty 
\end{cases},
\]

which, however, is not bijective and not coercive.

Then we have local ill-posedness at \( x^\dagger \) if \( x^\dagger \leq -1 \) or \( x^\dagger \geq 1 \). At \( x^\dagger = 0 \) we have local well-posedness for all \( \kappa > 0 \) due to a local uniform monotonicity condition with \( \zeta(t) = t^{\kappa+1} \) such that

\[
\| x - x^\dagger \| \leq \| F(x) - F(x^\dagger) \|^{1/\kappa} \quad \text{for all} \quad x \in B_1(0). \quad (CSE)
\]

Lavrentiev regularization allows for linear convergence if \( \kappa = 1 \), Hölder convergence rates for \( \kappa > 1 \), and there even occurs a superlinear convergence rate at \( x^\dagger = 0 \) if \( 0 < \kappa < 1 \).
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If \((**)\) is **locally ill-posed** at \(x^\dagger\), then \(\alpha(\delta) \sim \delta\) is not appropriate for Lavrentiev regularization, and the decay of \(\alpha(\delta)\) as \(\delta \to 0\) has to be slower in order to ensure convergence.

We have for the **bias** \(\|x_\alpha - x^\dagger\|\) with \(F(x_\alpha) + \alpha(x_\alpha - \bar{x}) = y\):

**Proposition**

\[
\lim_{\alpha \to 0} \|x_\alpha - x^\dagger\| = 0 \quad \text{if and only if} \quad x^\dagger = x_{mn}^\dagger,
\]

\[
\|x_\delta^\alpha - x^\dagger\| \leq \|x_\alpha - x^\dagger\| + \frac{\delta}{\alpha}.
\]

Consequently, an a priori choice \(\alpha(\delta)\) satisfying

\[
\alpha(\delta) \to 0 \quad \text{and} \quad \frac{\delta}{\alpha(\delta)} \to 0 \quad \text{as} \quad \delta \to 0
\]

yields **convergence** \(\lim_{\delta \to 0} \|x_\alpha(\delta) - x_{mn}^\dagger\| = 0\).
Under nonlinearity conditions on \( F \) Hölder convergence rates

\[
\| x_\alpha^\delta - x^\dagger \| = O \left( \delta^{\frac{p}{p+1}} \right) \quad \text{as} \quad \delta \to 0,
\]

occur for \( 0 < p \leq 1 \) with \( A := F'(x^\dagger) \) and \( \alpha(\delta) \sim \delta^{\frac{1}{p+1}} \) with the power-type source condition

\[
x^\dagger - \bar{x} = A^p w, \quad w \in X,
\]

where the fractional powers \( A^p \) of the monotone linear operator \( A \) are defined by a Dunford integral as

\[
A^p v := \frac{\sin(p\pi)}{\pi} \int_0^\infty s^{p-1}(A + sl)^{-1}Av \, ds, \quad v \in X.
\]

See, e.g., TAUTENHAHN 2002.
Let us consider a variational source condition adapted to the monotonicity structure:

Variational source condition (VSC-Lav)

We assume to have a constant $0 \leq \beta < 1$ and an index function $\varphi$ such that for all $x \in \mathcal{M}$

$$\langle x^\dagger - \bar{x}, x^\dagger - x \rangle \leq \beta \|x^\dagger - x\|^2 + \varphi(\langle F(x) - F(x^\dagger), x - x^\dagger \rangle).$$

Here, $\varphi$ must be an index function with $\lim_{t \to +0} \frac{\varphi(t)}{\sqrt{t}} \geq c > 0$.

The next theorem is proved by the general Young inequality

$$ab \leq \int_0^a f(t) \, dt + \int_0^b f^{-1}(t) \, dt$$

for $a, b \geq 0$ and an index function $f$. 
Theorem

Let under the standing assumptions \((VSC - Lav)\) be satisfied for the \(\tilde{x}\)-minimum-norm solution \(x^\dagger = x^\dagger_{mn}\) of \((**\)) with \(\mathcal{M}\) such that for the choice of \(\alpha > 0\) all regularized solutions \(x^\delta_\alpha\) from \((Lav)\) belong to \(\mathcal{M}\) for sufficiently small \(\delta > 0\). Then one has the estimate

\[
\|x^\delta_\alpha - x^\dagger\|_2^2 \leq \frac{1}{(1 - \beta)^2} \frac{\delta^2}{\alpha^2} + \frac{2}{1 - \beta} \Psi(\alpha),
\]

for such \(\delta > 0\), where \(\Psi(\alpha)\) is introduced as follows:

Let \(f\) be an index function such that its antiderivative \(\tilde{f}(s) := \int_0^s f(t)dt\) satisfies the condition \(\tilde{f}(\varphi(s)) \leq s\) for \(s > 0\) and let \(G(\alpha) \geq \int_0^\alpha f^{-1}(\tau)d\tau\). Then we set \(\Psi(\alpha) := \frac{G(\alpha)}{\alpha}\).
Moreover, for the a priori choice

$$\alpha(\delta) \sim \Theta^{-1}(\delta^2)$$

with $\Theta(\alpha) := \alpha^2 \psi(\alpha)$, which satisfies

$$\alpha(\delta) \to 0 \quad \text{and} \quad \frac{\delta}{\alpha(\delta)} \to 0 \quad \text{as} \quad \delta \to 0,$$

this yields the convergence rate

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| = O \left( \frac{\delta}{\Theta^{-1}(\delta^2)} \right) = O \left( \sqrt{\psi(\Theta^{-1}(\delta^2))} \right).$$

The best possible rate occurs for $\varphi(t) \sim \sqrt{t}$ and $\psi(t) \sim t$ as

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| = O(\delta^{\frac{1}{3}}) \quad \text{if} \quad \alpha(\delta) \sim \delta^{\frac{2}{3}}.$$
Logarithmic type variational source conditions

We consider $(VSC - Lav)$ with

$$\varphi(t) = \frac{1}{-\ln t}, \quad f(t) = e^{-\frac{1}{t}} \leq \frac{1}{t^2} e^{-\frac{1}{t}} = \varphi^{-1}(t), \quad f^{-1}(t) = \frac{1}{-\ln t},$$

$$G(\alpha) = \alpha \frac{1}{-\ln \alpha} \geq \int_{0}^{\alpha} \frac{1}{-\ln t} dt, \quad \Theta(t) \sim t^2 \frac{1}{-\ln t}, \text{ for } \alpha, t \in (0, 1).$$

This yields the logarithmic rate $O\left(\sqrt{\psi(\Theta^{-1}(\delta^2))}\right)$.

Since the a priori choice $\alpha(\delta) \sim \Theta^{-1}(\delta^2)$ cannot be determined explicitly in this logarithmic case, a more convenient choice is $\alpha(\delta) \sim \sqrt{\delta}$ which implies

$$\|x_{\alpha(\delta)} - x^\dagger\| = O\left(\frac{1}{\sqrt{-\ln(\delta)}}\right).$$
Hölder type variational source conditions

For exponents $\mu \in (0, \frac{1}{2}]$, we consider $(VSC - Lav)$ with

$$\varphi(t) = t^\mu, \quad f(t) = \frac{1}{\mu} t^{\frac{1-\mu}{\mu}}, \quad f^{-1}(t) = (\mu t)^{\frac{\mu}{1-\mu}},$$

$$\tilde{f}(s) = s^{\frac{1}{\mu}}, \quad G(\alpha) = (1 - \mu) \mu^{\frac{\mu}{1-\mu}} \alpha^{\frac{1}{1-\mu}}, \quad \Phi(t) \sim t^{\frac{2-\mu}{1-\mu}}.$$

According to the theorem this yields the Hölder rate

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| = O(\delta^{\frac{\mu}{2-\mu}}) \quad \text{if} \quad \alpha(\delta) \sim \delta^{\frac{2(1-\mu)}{2-\mu}}.$$

Note that $O \left( \delta^{\frac{\mu}{2-\mu}} \right) = O \left( \delta^{\frac{p}{2p+1}} \right)$ for $\mu := \frac{2p}{2p+1}$, $0 < p \leq \frac{1}{2}$. 
For linear forward operators $F := A \in \mathcal{L}(X, X)$ we outline situations of relations to the variational source conditions:

**Selfadjoint A:** First let $A = A^*$, $x^\dagger = A^p w$, $\bar{x} := 0$ and $0 < p \leq \frac{1}{2}$.

$$\langle x^\dagger, x \rangle = \langle w, A^p x \rangle \leq \|w\| \|A^p x\|$$
$$\leq \|w\| \|x\|^{1-2p} \|A^{1/2} x\|^{2p} = \|w\| \|x\|^{1-2p} \langle A x, x \rangle^p$$

based on the interpolation inequality. By using Young's ineq. $ab \leq a^\xi \eta + b^{n_\eta}$ with $a = \|x\|^{1-2p}$, $b = \|w\| \langle A x, x \rangle^p$, $\xi = \frac{2}{1-2p}$ and $\eta = \frac{2}{1+2p}$ this yields the variational source condition

$$\langle x^\dagger, x \rangle \leq \left(\frac{1}{2} - p\right) \|x\|^2 + \left(\frac{1}{2} + p\right) \|w\|^{\frac{2}{2p+1}} \langle A x, x \rangle^{\frac{2p}{2p+1}} \quad \forall x \in X$$

with exponent $0 < \mu = \frac{2p}{2p+1} \leq \frac{1}{2}$, hence the Hölder rate

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| = O(\delta^{\frac{\mu}{2-\mu}}) = O(\delta^{\frac{p}{p+1}}) \quad \text{if} \quad \alpha(\delta) \sim \delta^{\frac{2(1-\mu)}{2-\mu}} = \delta^{\frac{1}{p+1}}.$$
Non-selfadjoint $A$: Secondly, we consider for $X = L^2(0, 1)$, the linear integration operator

$$[Ax](s) := \int_0^s x(t) \, dt, \quad 0 \leq s \leq 1,$$

which is monotone, but not selfadjoint, i.e. $A \neq A^*$. We have

$$\langle Ax, x \rangle = \frac{1}{2} \left( \int_0^1 x(t) \, dt \right)^2 \geq 0,$$

and for $x^\dagger \equiv 1$, with $x^\dagger \notin \mathcal{R}(A)$ and $x^\dagger \notin \mathcal{R}(A^*)$,

$$\langle x^\dagger, x \rangle = \int_0^1 x(t) \, dt \leq \sqrt{2} \langle Ax, x \rangle^{1/2} \quad \forall x \in X,$$

thus the convergence rate for the Lavrentiev regularization

$$\| x^\delta - x^\dagger \| = O \left( \delta^{1/3} \right) \quad \text{for} \quad \alpha \sim \delta^{2/3}.$$
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Let $A : X \to X$ be a bounded linear monotone operator. For

$$Ax = y \quad (\ast)$$

the linear Lavrentiev regularization acts as

$$x_\alpha^\delta = (A + \alpha I)^{-1}(y^\delta + \alpha \bar{x}).$$

**Proposition**

The linear equation $(\ast)$ is **locally well-posed everywhere** iff $A$ is continuously invertible, i.e. there is $K > 0$ such that

$$\|(A + \alpha I)^{-1}\| \leq K < \infty \quad \text{for all} \quad \alpha > 0,$$

where $K = \|A^{-1}\|$ holds true, but $(\ast)$ is **locally ill-posed everywhere** iff $\mathcal{N}(A) \neq \{0\}$ or $\mathcal{R}(A) \neq \overline{\mathcal{R}(A)}$. Then we have

$$\|(A + \alpha I)^{-1}\| = \frac{1}{\alpha} \quad \text{for all} \quad \alpha > 0.$$
For the maximal best possible error

\[ E_{x^\dagger}(\delta) := \sup_{y^\delta \in X: \|y - y^\delta\| \leq \delta} \inf_{\alpha > 0} \| x^\delta_\alpha - x^\dagger \| \]

of Lavrentiev regularization to the linear equation \((*)\) we have

\[ E_{x^\dagger}(\delta) = O(\delta) \quad \text{as} \quad \delta \to 0 \]

in the well-posed case, but

\[ E_{x^\dagger}(\delta) = o(\sqrt{\delta}) \quad \text{as} \quad \delta \to 0 \quad \text{implies} \quad x^\dagger - \bar{x} = 0 \]

in the ill-posed case.